ON NON-RIGID DEL PEZZO FIBRATIONS OF LOW DEGREE

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Abstract. We consider $\mathbb{P}(1,1,1,2)$ bundles over $\mathbb{P}^1$ and construct hypersurfaces of these bundles which form a degree 2 del Pezzo fibration over $\mathbb{P}^1$ as a Mori fibre space. We classify all such hypersurfaces whose type III or IV Sarkisov links pass to a different Mori fibre space. A similar result for cubic surface fibrations over $\mathbb{P}^2$ is also presented.

Contents

1. Introduction

One possible outcome of the minimal model program is a Mori fibre space.

Definition 1.1. A *Mori fibre space* is a contraction $\varphi\colon X \to S$, where

(1) X is $\mathbb{Q}$-factorial with at worst terminal singularities,
(2) $-K_X$ is φ-ample,
(3) $\rho(X) = \rho(S) + 1$,
(4) $\dim S < \dim X$.

Of course, by definition above, there are three cases of 3-dimensional Mori fibre spaces:

(i) X is a Fano 3-fold, when $\dim S = 0$,
(ii) X is a del Pezzo fibration, when $\dim S = 1$,
(iii) X is a conic bundle, when $\dim S = 2$.

Definition 1.2. Let $\varphi\colon X \to S$ and $\varphi'\colon X' \to S'$ be Mori fibre spaces such that there is a birational map $f\colon X \dashrightarrow X'$. The map f is said to be *square* if there is a birational map $g\colon S \dashrightarrow S'$, which makes the diagram

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, "\varphi"'] & X' \arrow[d, "\varphi'"] \\ S \arrow[r, dashed, "g"] & S' \end{tikzcd}$$

commute and, in addition, the induced birational map $f_L\colon X_L \to X'_L$ between the generic fibres is biregular. In this situation, we say that the two Mori fibre spaces $X \to S$ and $X' \to S'$ are *birational square.*

Definition 1.3. A Mori fibre space $X \to S$ is *birationally rigid* if for any birational map $f\colon X \dashrightarrow X'$ to another Mori fibre space $X' \to S'$, there exists a birational selfmap $\alpha\colon X \dashrightarrow X$ such that the composite $f \circ \alpha\colon X \dashrightarrow X'$ is square.

In [5] it was shown that a general member in the list of 95 families of Fano 3-folds is birationally rigid. Birational rigidity of conic bundles has been studied by a number of people, for example see [13], [14], [22], [23] and [3]. Del Pezzo fibrations split into 9 cases according to the degree of the fibres, that is the intersection number K_L^2, where L is the generic fibre. If the degree is greater than 5, it is known that the 3-fold is rational. Alexeev in [1] proved that a standard degree 4 del Pezzo fibration is birational to a conic bundle, and hence they are non-rigid. Rigidity of degree 3 del Pezzo fibrations have been studies by many authors; for example see [19] and [3]. Birational geometry of lower degree del Pezzo fibrations has been only studied in the smooth case. The main contributions being works of Pukhlikov [19] and Grinenko [8–10]. In fact the smoothness condition of these varieties is very restrictive as in many cases the 3-fold X has nonsmooth terminal singularities. In that regard most of the families constructed in this article have index 2 singularities.

We provide a natural construction for degree 2 del Pezzo fibrations, denoted by dP_2. This is followed by classifying those which admit another Mori fibre space as a birational model (not birational square) where the other model is obtained by restriction of the 2-ray game of the ambient space on the 3-fold. In particular, these 3-folds are non-rigid.

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2. Construction

Definition 2.1. A *weighted bundle over* $\mathbb{P}^n$ is a rank 2 toric variety $\mathcal{F} = TV(A, I)$ defined by

(i) $\mathrm{Cox}(\mathcal{F}) = \mathbb{C}[x_0, \dots, x_n, y_0, \dots, y_m]$.
(ii) The irrelevant ideal of $\mathcal{F}$ is $I = (x_0, \dots, x_n) \cap (y_0, \dots, y_m)$.

(iii) and the $(\mathbb{C}^*)^2$ action on $\mathbb{C}^{n+m+2}$ is given by

$$A = \begin{pmatrix} 1 & \dots & 1 & -\omega_0 & -\omega_1 & \dots & -\omega_m \\ 0 & \dots & 0 & 1 & a_1 & \dots & a_m \end{pmatrix},$$

where ω_i are non-negative integers and $\mathbb{P}(1, a_1, \dots, a_m)$ is a weighted projective space.

Definition 2.2. (a) Let T be a rank 2 toric variety. Suppose t is a generating variable in the Cox ring of T and that the action of the $(\mathbb{C}^*)^2$ on t is given by $t \mapsto \lambda^a \mu^b t$, where $(a, b) \in \mathbb{Z}^2 \backslash \{(0, 0)\}$. We say that the number $\frac{a}{b}$ is the *ratio weight* of the variable t. Note that the ratio weight could be a rational number or $\infty = \frac{|a|}{0}$ or $-\infty = \frac{-|a|}{0}$.

(a) Let T be a rank 2 toric variety with $\operatorname{Cox}(T) = \mathbb{C}[t_1, \dots, t_k]$. Define a total order $\preceq$ on $\{t_0, \dots, t_k\}$ by $t_i \preceq t_j$ if and only if the ratio weight of t_j is less than or equal to the ratio weight of t_i. Note that we allow $-\infty$ and ∞ in their own right. If the ratio weight of t_i is strictly bigger than the one for t_j, we write $t_i \prec t_j$.

Remark 2.3. Note that the order $\preceq$ above is induced by the usual order in the set of extended real numbers in the **reverse** direction!

Without loss of generality we can assume the variables of the $\operatorname{Cox}(\mathcal{F})$ in Definition 2.1 are in order with respect to $\preceq$. Let $Y_0, \dots, Y_r$ be the partition of $y_0, \dots, y_m$ such that variables contained in each Y_i have the same ratio weight and that Y_i is nonempty and contains all variables with that ratio weight. Furthermore we assume that they are in order with $Y_i \prec Y_{i+1}$, meaning the ratio weight of the variable in Y_i is strictly bigger than the ratio weight of variables in Y_{i+1}. Note that this last condition makes $Y_0, \dots, Y_r$ a unique partition of $y_0, \dots, y_m$.

Consider the ideal $I_j = (x_0, \dots, x_n, Y_0, \dots, Y_{j-1}) \cap (Y_j, \dots, Y_r) \subset \operatorname{Cox}(\mathcal{F})$. Let $\mathcal{F}_j$ be the rank two toric variety defined by $TV(A, I_j)$, i.e.

$$\mathcal{F}_j = (\mathbb{C}^{n+m+2} \backslash V(I_j)) /\!\!/ (\mathbb{C}^*)^2$$

in particular $\mathcal{F}_0 = \mathcal{F}$. The following is an observation of the Theorem 4.1 in [4], also known in [21].

Theorem 2.4. *Let $\mathcal{F}/\mathbb{P}^n$ be a weighted bundle as before. Then the 2-ray link of $\mathcal{F}$ is given by one of the following:*

(1) If $|Y_r| = 1$, i.e. the set Y_r has only one element, then

$$\begin{tikzcd} & \mathcal{F}_0 \arrow[r, "\Psi_1"] \arrow[dl, "\Phi"'] & \mathcal{F}_1 \arrow[r, "\Psi_2"] & \cdots \arrow[r, "\Psi_{r-1}"] & \mathcal{F}_{r-1} \arrow[dr, "\Phi'"] & \\ \mathbb{P}^1 & & & & & \mathcal{F}_r \end{tikzcd}$$

where $\mathcal{F}_0 = \mathcal{F}$, Ψ_i are isomorphisms in codimension one and Φ' is a divisorial contraction.

(2) If $|Y_r| > 1$, then

$$\begin{tikzcd} & \mathcal{F}_0 \arrow[r, "\Psi_1"] \arrow[dl, "\Phi"'] & \mathcal{F}_1 \arrow[r, "\Psi_2"] & \cdots \arrow[r, "\Psi_r"] & \mathcal{F}_r \arrow[dr, "\Phi'"] & \\ \mathbb{P}^1 & & & & & \mathbb{P} \end{tikzcd}$$

where $\mathcal{F}_0 = \mathcal{F}$, Ψ_i are isomorphisms in codimension one, Φ' is a fibration and $\mathbb{P} = \mathbb{P}(a_{r_1}, \ldots, a_{r_k})$, where $a_{r_1}, \ldots, a_{r_k}$ are the denominators of the ratio weights of the variables in Y_r.

Note that case (1) in this theorem is the Type III Sarkisov link of $\mathcal{F}$ and case (2) is the Type IV.

Definition 2.5. Let $\mathcal{F}/\mathbb{P}^n$ be a weighted bundle as in Definition 2.1, and $\mathcal{F}_i$ be the varieties appearing in its 2-ray link of Theorem 2.4. Let $\overline{X}\colon (f = 0) \subset \mathbb{C}^{n+m+2}$ be a hypersurface in $\mathbb{C}^{n+m+2}$, the Cox cover of $\mathcal{F}$, defined by $f \in \mathbb{C}[x_0, \ldots, x_n, y_0, \ldots, y_m]$. Assume f is irreducible, reduced and homogeneous with respect to the action of $(\mathbb{C}^*)^2$. Define $X_i \subset \mathcal{F}_i$ to be

$$X_i = (\overline{X} \backslash V(I_i))/(\mathbb{C}^*)^2$$

and let ψ_i (respectively φ, φ') be the restriction of Ψ_i (respectively Φ, Φ') to X_{i-1}. Then we say X_0 has an *$\mathcal{F}$-link* if

(i) ψ_i are isomorphisms in codimension one (possibly isomorphisms).
(ii) φ and φ' are extremal contractions.

In other words, X_0 has an $\mathcal{F}$-link if the 2-ray game of X_0 is obtained by the restriction of the 2-ray game of $\mathcal{F}_0$ (although some φ_i may be isomorphisms and hence redundant from the game). If in addition, each X_i is $\mathbb{Q}$-factorial with terminal singularities, then we say X_0 has an *$\mathcal{F}$-Sarkisov link.*

3. Sarkisov links from general $dP_2/\mathbb{P}^1$ hypersurfaces

We consider weighted bundles over $\mathbb{P}^1$ with fibre $\mathbb{P}(1,1,1,2)$; these are a natural place to embed 3-fold degree 2 del Pezzo fibrations.

Definition 3.1. A 3-fold X is a *degree 2 del Pezzo fibration over* $\mathbb{P}^1$ (denoted by dP_2 fibration, or simply $dP_2/\mathbb{P}^1$) if X has an extremal contraction of fibre type $\varphi\colon X \to \mathbb{P}^1$ such that

(a) X has at worst terminal singularities and is $\mathbb{Q}$-factorial.
(b) The nonsingular fibres of φ are del Pezzo surfaces of degree two.

Let $\mathcal{F}$ be a rank two toric variety defined by $\mathcal{F} = TV(I, A)$, where $I \subset \mathbb{C}[u, v, x, y, z, t]$ is the irrelevant ideal $I = (u, v) \cap (x, y, z, t)$ and A is the representing matrix of the action of $\mathbb{C}^* \times \mathbb{C}^*$ given by

$$A = \begin{pmatrix} 1 & 1 & -\alpha & -\beta & -\gamma & -\delta \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \quad . \tag{1}$$

Remark 3.2. Up to the action of $\mathrm{SL}(2,\mathbb{Z})$, any matrix of type (1) can be written uniquely in one of the following forms:

$$(i) \qquad A = \begin{pmatrix} 1 & 1 & 0 & -a & -b & -c \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \qquad 0 < c\,,\, 0 \le a \le b$$

$$(ii) \qquad A = \begin{pmatrix} 1 & 1 & -a & -b & -c & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \qquad 0 \le a \le b \le c$$

$$(iii) \qquad A = \begin{pmatrix} 1 & 1 & -a & -b & -c & -1 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \qquad 0 < a \le b \le c \quad .$$

The Picard group of $\mathcal{F}$ is isomorphic to $\mathbb{Z}^2$. Let L and M be Weil divisors of $\mathcal{F}$ with weights $(1,0)$ and $(0,1)$. For example in the case (i) above $u \in H^0(\mathcal{F}, L)$ and $x \in H^0(\mathcal{F}, M)$. A simple toric singularity analysis shows that $\mathcal{F}$ is smooth away from the curve $\Gamma_t = (x = y = z = 0)$. The curve Γ_t is a rational curve with singularity of transverse type $\frac{1}{2}(1,1,1)$ along Γ_t.

Let $D = 4M - eL \in \mathrm{Div}(\mathcal{F})$ be a divisor in $\mathcal{F}$ and $X = (f = 0) \subset \mathcal{F}$ be the hypersurface of $\mathcal{F}$ defined by a general $f \in H^0(\mathcal{F}, D)$. We say that $X \subset \mathcal{F}$ has bi-degree $(-e, 4)$ and encode these information about X and $\mathcal{F}$ with the notation

$$\begin{pmatrix} -e \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & -\alpha & -\beta & -\gamma & -\delta \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \quad .$$

The goal is to find conditions on X and $\mathcal{F}$ such that X is a Mori fibre space, whose generic fibre is a del Pezzo surface of degree 2, that has an $\mathcal{F}$-Sarkisov link to another Mori fibre space.

3.1. **The main result.**

Theorem 3.3. *Consider a hypersurface $X \subset \mathcal{F}$ with*

$$\begin{pmatrix} -e \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & -\alpha & -\beta & -\gamma & -\delta \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \quad ,$$

where the weights α, β, γ are normalised with $\gamma \ge \beta \ge \alpha \ge 0$ and $\delta \ge 0$. Suppose the Type III *or* IV *2-ray game of $\mathcal{F}$ restricts to a Sarkisov link for X. Then the weights $\alpha, \beta, \gamma, \delta, e$ are among those appearing in the left-hand column of Table 1.*

Moreover, we show in 4.1 below that if X is a general hypersurface of type $(\alpha, \beta, \gamma, \delta; e)$ from table 1, then X is nonrigid. The Sarkisov link to another Mori fibre space is described in the remaining columns of Table 1.

4. General hypersurfaces

In this section, we prove the constructive part, the second part, of the Theorem 3.3 in one direction by calculating the birational link for a general hypersurface in each family in Theorem 3.3

No.	$(\alpha,\beta,\gamma,\delta;e)$	ψ_1	ψ_2	φ'	*new model*
1	$(0,0,0,0;-1)$	*n/a*	*n/a*	*contraction*	$\mathbb{P}(1,1,1,2)$
2	$(0,0,0,1;0)$	*n/a*	*n/a*	*contraction*	$Y_4\subset\mathbb{P}(1,1,1,2,2)$
3	$(0,0,1,0;0)$	*n/a*	*n/a*	*contraction to a line*	$Y_4\subset\mathbb{P}(1,1,1,1,2)$
4	$(0,1,1,0;0)$	*flop of* $2\times\mathbb{P}^1$	*n/a*	*fibration*	dP_2 *fibration*
5	$(0,0,1,1;0)$	*flop of 4 disjoint* $\mathbb{P}^1$	*n/a*	*divisorial contraction to a point*	$Y_4\subset\mathbb{P}^4$
6	$(1,1,1,1;2)$	$\cong$	*n/a*	*fibration*	*conic bundle with discriminant* $\Delta_8\subset\mathbb{P}^2$
7	$(0,1,1,1;1)$	*flop*	*flip*	*fibration*	dP_3 *fibration*
8	$(0,1,1,2;2)$	*flop*	*n/a*	*fibration*	*conic bundle over* $\mathbb{P}(1,1,2)$ *with disc.* $\Delta_{10}\subset\mathbb{P}(1,1,2)$
9	$(0,1,2,1;2)$	*flop*	$\cong$	*contraction*	$Y_6\subset\mathbb{P}(1,1,1,2,3)$
10	$(0,1,1,3;3)$	*flop*	*n/a*	*contraction*	$Y_6\subset\mathbb{P}(1,1,2,2,3)$
11	$(0,2,2,1;2)$	*anti-flip*	$\cong$	*fibration*	dP_1 *fibration*
12	$(0,1,2,3;3)$	*anti-flip*	*flop*	*contraction*	$Y_5\subset\mathbb{P}(1,1,1,1,2)$
13	$(0,1,2,4;4)$	*anti-flip*	$\cong$	*fibration*	dP_2 *fibration over* $\mathbb{P}(1,2)$

TABLE 1. Data of Type III and IV links from general degree 2 del Pezzo hypersurface fibrations

and then we show in subsection 4.3 that these hypersurfaces are indeed $dP_2/\mathbb{P}^1$. These links are provided from the restriction of the natural 2-ray game of the ambient toric variety $\mathcal{F}$ to X.

4.1. **Geometry of the links.** In order to match the notation of Theorem 2.4, in each case we rewrite the defining numerical system, normalised by the order $\preceq$, and give the numerical system of the rank 2 variety at the end of each link. Rather than following the order in Table 1, we analyse cases together according to the structures at the end of their links.

4.1.1. *Links to conic bundles.*

Family 6: $u=v\prec t\prec x=y=z$

The 2-ray game of $\mathcal{F}$ starts by Ψ_1, which is a flip of type $(2,2,-1,-1,-1)$ in the neighbourhood $(t\neq 1)$ of the flipping curve $\mathbb{P}^1_{u:v}$. The second and final step of the 2-ray game is a $\mathbb{P}^2$ fibration to $\mathbb{P}^2_{x:y:z}$. Considering X of bi-degree $(-2,4)$, the Newton polygon of X is

deg of u,v coefficient	
0	t^2
1	$tx^2\quad txy\quad\ldots\quad tyz\quad tz^2$
2	$x^4\quad x^3y\quad\ldots\quad yz^3\quad z^4$.

This means that f, the defining polynomial of X, includes terms of the form t^2 and $l(u,v)tx^2$ and $q(u,v)x^4$, where $l(u,v)$ is a general linear form in u,v and $q(u,v)$ is a general quadratic. We use the notation $t^2\in F$ to say that the monomial t^2 appears as a term of f. It is also

useful for us to describe f as the product of the following matrices:

$$\begin{pmatrix} u & v & t \end{pmatrix} \begin{pmatrix} *_4 & *_4 & *_2 \\ *_4 & *_4 & *_2 \\ *_2 & *_2 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ t \end{pmatrix} \quad , \tag{2}$$

where by $*_k$ we mean a general homogeneous polynomial of degree k in variables x, y, z.

Having the monomial $t^2 \in f$ ensures that X does not intersect with the singular locus of $\mathcal{F}$ as $\mathrm{Sing}(\mathcal{F}) = \Gamma_t$. Having this key monomial also shows that ψ_1, the restriction of Ψ_1 to X, is an isomorphism on X. The restriction of Φ' to X defines a fibration to $\mathbb{P}^2_{x:y:z}$ with fibres being conic curves. The discriminant of this conic is the determinant of the 3×3 matrix in (2). The degree of the discriminant in this case is 8.

Family 8: $u = v \prec x \prec y = z = t$

Let us describe the birational geometry of the ambient space $\mathcal{F}$. The 2-ray game of $\mathcal{F}$ starts by mapping to $\mathbb{P}^1$ in one side (the given extremal contraction) and anti-flip $(1, 1, -1, -1, -2)$ in the other side. This anti-flip can be read by fixing the action of the second component of the $(\mathbb{C}^*)^2$ in the neighbourhood $(x \neq 0)$ by putting $x = 1$. Then the game follows by an extremal contraction of fibre type to $\mathbb{P}(1, 1, 2)$. To restrict this toric 2-ray game to X, we need to know f, the defining polynomial of X, which can be seen from the Newton polygon of X,

deg of u, v coefficient						
0		x^2t	xy^2	xyz	xz^2	
1	xyt	xzt	xy^3	xy^2z	xyz^2	xz^3
2		y^2t	yzt	z^2t	t^2	.

Here our essential terms in f are x^2t and $q(u, v)t^2$, where $q(u, v)$ is a general quadratic in u, v. Having $q(u, v)t^2 \in f$ means that the singular locus of (a general quasismooth) X is the intersection of X with Γ_t, which in this case is only two points $(q = 0) \cap \Gamma_t$.

The $\mathcal{F}$-Sarkisov link of a general X in this family, starts by an Atiyah flop and follows by a fibration to $\mathbb{P}(1, 1, 2)$ with conic curve fibres. The flop is the restriction of the $(1, 1, -1, -1, -2)$ anti-flip on $\mathcal{F}$. The restriction is a flop because the monomial $x^2t \in f$ allows us to eliminate the variable t in the neighbourhood $(x \neq 0)$.

Similar to the previous case, considering the defining polynomial of X in the form

$$\begin{pmatrix} u & v & t \end{pmatrix} \begin{pmatrix} *_4 & *_4 & *_3 \\ *_4 & *_4 & *_3 \\ *_3 & *_3 & *_2 \end{pmatrix} \begin{pmatrix} u \\ v \\ t \end{pmatrix} \tag{3}$$

tells us that the degree of the discriminant of the conic in this case is 10.

Remark 4.1. In [17], a list of possible singularities that the base variety of a conic bundle can admit is provided. By Theorem 1.2.7. in [17], $\mathbb{P}(1,1,2)$ is a legal base since it has only a quotient singularity $\frac{1}{2}(1,1)$, which is Du Val.

4.1.2. *Links to del Pezzo fibrations.*

Family 4: $u = v \prec x = t \prec y = z$

The 2-ray game of $\mathcal{F}$ in this case is represented by

$$\begin{tikzcd} & \mathcal{F} \arrow[dl, "\Phi"'] \arrow[dr, "\Psi_1^-"] & & \mathcal{F}_1 \arrow[dl, "\Psi_1^+"'] \arrow[dr, "\Phi'"] & \\ \mathbb{P}^1_{u:v} & & \mathcal{G} & & \mathbb{P}^1_{y:z} \end{tikzcd},$$

where the composition map $\Psi_1 = (\Psi_1^+)^{-1} \circ \Psi_1^-$, is a toric 4-fold flop. Both Ψ_1^- and Ψ_1^+ are isomorphism away from $\mathbb{P}^1 \times \mathbb{P}^1$. The first map, Ψ_1^-, contracts the surface $\mathbb{P}^1_{u:v} \times \mathbb{P}_{x:t}(1,2)$ to $\mathbb{P}^1_{x:t}$ and Ψ_1^+ contracts $\mathbb{P}^1_{y:z} \times \mathbb{P}^1_{x:t}$ to the same line. This composition defines Ψ_1 as a toric 4-fold flop. The next step of the 2-ray game, Φ' provides a fibration to $\mathbb{P}^1_{y:z}$ with fibres isomorphic to $\mathbb{P}(1,1,1,2)$.

The defining equation of X has the form $f = g + h$, where $g = g(x,t)$ is a quartic in variables x and t only. This ensures that the restriction of Ψ_1^- contracts two disjoint $\mathbb{P}^1$, defined by $(g = 0) \cap \mathbb{P}^1_{u:v} \times \mathbb{P}_{x:t}(1,2)$ to two points in $\mathbb{P}^1_{x:t}$, namely the solutions of $(g = 0) \subset \mathbb{P}(1,2)$. This argument shows that ψ_1 is formed of a flop $\psi_1 \colon X \to X_1$, which flops two disjoint copies of $\mathbb{P}^1$. At the end of the link, the restriction of Φ' to X_1 provides the extremal contraction of fibre type to $\mathbb{P}^1$ with degree 2 del Pezzo fibres.

Family 7: $u = v \prec x \prec t \prec y = z$

This case is similar to the previous one and the result was already found in [3]. A full analysis is given in [3] Family 5, §4.4.2..

Family 11: $u = v \prec x \prec t \prec y = z$

The diagram of the 2-ray game of $\mathcal{F}$ is

$$\begin{tikzcd} & \mathcal{F} \arrow[dl, "\Phi"'] \arrow[r, "\Psi_1"] & \mathcal{F}_1 \arrow[r, "\Psi_2"] & \mathcal{F}_2 \arrow[dr, "\Phi'"] & \\ \mathbb{P}^1_{u:v} & & & & \mathbb{P}^1_{y:z} \end{tikzcd},$$

where Ψ_1 is the anti-flip $(1,1,-1,-2,-2)$ flipping a copy of $\mathbb{P}^1$ to $\mathbb{P}(1,2,2)$. In particular, the flipping locus of $\mathcal{F}_1$ has line of singularity of transverse type $\frac{1}{2}(1,1,1)$. Note that $\mathcal{F}$ contains a singular line Γ_t, which is preserved by Ψ_1. The second anti-flip Ψ_2, is of type $(2,2,1,-3,-3)$, which flips a surface $\mathbb{P}(1,2,2)$ (including Γ_t) to a singular curve of transverse type $\frac{1}{3}(1,2,2)$. $\Phi' \colon \mathcal{F}_2 \to \mathbb{P}^1$ is a fibration, with $\mathbb{P}(1,1,2,3)$ fibres.

Now we consider the restriction of this game to X. The essential monomials of the defining polynomial of X are t^2 and x^3y. The first monomial, t^2 shows that $\Gamma_t \cap X$ is empty

for a general X. In fact, Bertini Theorem implies that X is smooth as the base locus of the linear system D includes only the curve $\Gamma_x = (u_0 : v_0; 1 : 0 : 0 : 0)$, which is guaranteed to be smooth by $x^3y \in f$.

The restriction of Ψ_1 to X is a Francia anti-flip as we can eliminate the variable y in a neighbourhood of the flipping curve using x^3y and implicit function theorem. Note that the variety X_1 has a $\frac{1}{2}(1,1,1)$ singularity obtained by this anti-flip. The restriction of Ψ_2 to X_1 is an isomorphism as $t^2 \in f$. And finally, $\varphi' \colon X_1 \to \mathbb{P}^1$ is a Mori fibre space with generic fibre isomorphic to a del Pezzo surface of degree 1.

Family 13: $u = v \prec x \prec y \prec z = t$

A similar argument shows that the general X in this case, after a Francia anti-flip has an extremal contraction of fibre type to $\mathbb{P}(1,2)$, with generic fibre isomorphic to a degree 2 del Pezzo surface.

4.1.3. *Links to Fano 3-folds.*

Family 1: $u = v \prec x = y = z = t$

The defining polynomial of a general X in this case is of the form $uf_4(x,y,z,t) = vg_4(x,y,z,t)$, for general degree 4 polynomials f and g in variables x, y, z, t. The 2-ray game of $\mathcal{F}$ is continued by a fibration Φ' to $\mathbb{P}(1,1,1,2)$ with $\mathbb{P}^1$ fibres. The restriction of this map to X provides $\varphi' : X \to \mathbb{P}(1,1,1,2)$, which contracts the divisor $(f = g = 0) \subset X$ to a curve in $\mathbb{P}(1,1,1,2)$, defined by the same set of equations.

Family 2: $u = v \prec x = y = z \prec t$

The 2-ray game of the ambient toric variety is described by

$$\begin{tikzcd} & \mathcal{F} \arrow[dl, "\Phi"'] \arrow[dr, "\Phi'"] & \\ \mathbb{P}^1_{u:v} & & \mathbb{P}\ (1,1,1,2,2) \end{tikzcd} ,$$

where Φ' is the divisorial contraction defined by the basis of the Riemann-Roch space of the divisor $D_x \sim (x = 0)$. More precisely, the equation of Φ' is

$$\Phi' \colon \mathcal{F} \to \mathbb{P}(1,1,1,2,2)$$

$$(u : v; x : y : z : t) \mapsto (x : y : z : ut : vt) \qquad .$$

It is clear from this equation that the divisor $(t = 0)$ is contracted to the surface $\mathbb{P}^2_{x:y:z}$. Note that this map has no base point, as the locus where all these monomials vanish is precisely the Cox irrelevant ideal of $\mathcal{F}$, i.e. $(u, v) \cap (x, y, z, t)$.

The equation of a general X in this family is of the form $t^2q(u,v) = f(x,y,z) + \dots$, where q is a quadratic polynomial in u, v and f is a quartic with variables x, y, z. Such X has two singular points of type $\frac{1}{2}(1,1,1)$, which are located at the intersection of X with Γ_t, that is the solutions of $(q = 0) \cap \Gamma_t$. Then X follows the 2-ray game of the ambient

space by contracting the divisor $(t=0)$ to the curve $(f=0) \subset \mathbb{P}^2_{x:y:z}$ on an index 3 Fano 3-fold defined by $X_4 \subset \mathbb{P}(1,1,1,2,2)$.

The equation of the Fano 3-fold, the image of X under this map, can be derived explicitly using this coordinate map. For example if the coordinate variables on $\mathbb{P}(1,1,1,2,2)$ are x, y, z, u', v', then this Fano variety is the hypersurface defined by

$$q(u',v') = f(x,y,z) + \dots \qquad .$$

Corollary 4.2. *An index 4 Fano 3-fold hypersurface $Y_4 \subset \mathbb{P}(1,1,1,2,2)$ is birational to a degree 2 del Pezzo fibration over $\mathbb{P}^1$.*

Family 3: $u = v \prec x = y = t \prec z$

Analysis of the link is similar to the previous case with the final divisorial contraction Φ' with equation

$$(u; v; x : y : t : z) \mapsto (uz : vz : x : y : t) \qquad .$$

The image of X under this map is an index 2 Fano hypersurface defined by a quartic in $\mathbb{P}(1,1,1,1,2)$.

Corollary 4.3. *An index 2 Fano 3-fold hypersurface $Y_4 \subset \mathbb{P}(1,1,1,1,2)$ is birational to a degree 2 del Pezzo fibration over $\mathbb{P}^1$.*

Family 5: $u = v \prec x = y \prec t \prec z$

The 2-ray game of $\mathcal{F}$ starts by a flop and continues by a divisorial contraction to $\mathbb{P}^4$. The toric flop contracts a copy of $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1$ and extracts another $\mathbb{P}^1 \times \mathbb{P}^1$. The restriction of this birational map to X flops 4 analytically disjoint copies of $\mathbb{P}^1$, since the defining polynomial of X includes a quartic in the x, y variables.

A general X in this family is singular at two points of type $\frac{1}{2}(1,1,1)$. As usual, these points are the locus where X meets Γ_t. In fact we can assume that the defining polynomial of X is of the form $(u^2+v^2)t^2 + f(x,y) + \dots$, where f is a general quartic in x, y. The divisorial contraction has the coordinate description

$$(u : v; x : y : t : z) \mapsto (uz^2 : vz^2 : xz : yz : t) \qquad ,$$

which shows that the divisor $(z=0)$ gets contracted to the point $p_t \in \mathbb{P}^4$. The equation near this point has a local type $u^2+v^2+x^4+y^4$. In other words this point is terminal. In fact this example was already known to be nonrigid. See [5], Example 7.5.1.

Family 9: $u = v \prec x \prec t \prec y \prec z$

The 2-ray game on the ambient space is

$$\begin{tikzcd} & \mathcal{F} \arrow[r, "\Psi_1"] \arrow[dl, "\Phi"'] & \mathcal{F}_1 \arrow[r, "\Psi_2"] & \mathcal{F}_2 \arrow[dr, "\Phi'"] & \\ \mathbb{P}^1_{u:v} & & & & \mathbb{P}(1,1,1,2,3) \end{tikzcd} ,$$

where Ψ_1 is the anti-flip $(1,1,-1,-1,-2)$ and Ψ_2 is the flip $(2,2,1,-1,-3)$. The final contraction is

$$\Phi' \colon (u:v;x:t:y:z) \mapsto (u_0:v_0:y:x_0:z_0) = (uz:vz:y:xz:tz) \qquad ,$$

which is the ordinary blow up of the smooth point $p_y \in \mathbb{P}(1,1,1,2,3)$. The Newton polygon of X in this family is described by

deg of u, v coefficient	
0	$t^2 \quad x^3z \quad x^2y^2 \quad xty$
1	$xy^3 \quad x^2yz \quad xtz \quad ty^2$
2	$y^4 \quad xy^2z \quad tyz \quad x^2z^2$
3	$xyz^2 \quad tz^2 \quad y^3z$
4	$xz^3 \quad y^2z^2$
5	yz^3
6	z^4 .

Having the term $t^2 \in f$, the defining polynomial of X, guarantees smoothness of X. The map ψ_1, the restriction of Ψ_1 to X, is an Atiyah flop as the variable z can be eliminated in a neighbourhood of the flopping curve using the monomial x^3z and the implicit function theorem. Similarly, we can observe that ψ_2 is an isomorphism as $t^2 \in f$. The image of X_1 under φ' is an index 2 Fano hypersurface Y defined by a degree 6 polynomial in $\mathbb{P}(1,1,1,2,3)$. One can see that under this map, the divisor $(z=0)$ goes to the point $p_y \in Y$. This point is a cA_1 point as the defining polynomial of Y is

$$t_0^2 + x_0^3 + y^4u_0v_0 + u_0^6 + v_0^6 + \dots \qquad .$$

Conversely, a general Fano hypersurface $Y_6 \subset \mathbb{P}(1,1,1,2,3)$ with a cA_1 point is birational to a degree 2 del Pezzo fibration over $\mathbb{P}^1$.

Family 10: $u = v \prec x \prec y = z \prec t$

The 2-ray game on $\mathcal{F}$ is

$$\begin{tikzcd} & \mathcal{F}_0 \arrow[r, "\Psi_1"] \arrow[dl, "\Phi"'] & \mathcal{F}_1 \arrow[dr, "\Phi'"] & \\ \mathbb{P}^1 & & & \mathbb{P}(1,1,2,2,3) \end{tikzcd} ,$$

where Ψ_1 is the anti-flip $(1,1,-1,-1,-3)$. And the final contraction is $\Phi'\colon \mathcal{F}_1 \to \mathbb{P}(1,1,2,2,3)$ defined by

$$(u; v; x : y : z : t) \mapsto (y : z : u_0 : v_0 : x_0) = (y : z : ut : vt : xt) \qquad .$$

This map contracts the divisor $(t = 0)$ on $\mathcal{F}_1$ to the line $\mathbb{P}^1_{y:z} \subset \mathbb{P}(1,1,2,2,3)$.

The Newton polygon of a general X in this family is

$\deg S^k(u,v,w)$	
0	$x^2t \quad xy^3 \quad xy^2z \quad xyz^2 \quad xz^3$
1	$y^4 \quad \ldots z^4 \quad xyt \quad xzt$
2	$y^2t \quad yzt \quad z^2t$
3	t^2 .

The coefficient of t^2 in the equation indicates that

$$\mathrm{Sing}(X) = \Gamma_t \cap X = 3 \times \frac{1}{2}(1,1,1) \quad .$$

The map ψ_1, obtained by restricting Ψ_1 to X is a flop $(1,1,-1,-1)$, as we are able to eliminate the variable t near the flopping curve using the monomial x^2t. The map φ' contracts the divisor $(t = 0) \subset X_1$ to the line $\mathbb{P}^1_{y:z}$ on an index 3 Fano variety Y defined by a degree 6 polynomial in $\mathbb{P}(1,1,2,2,3)$. The defining polynomial of Y is

$$x_0^2 + g_3(u_0, v_0) + uq_4(y,z) + vq_4'(y,z) + \ldots \qquad ,$$

where g_3 is a general cubic in the variables u_0, v_0; q and q' are general quartics in y, z. Hence Y is smooth along $\mathbb{P}^1_{y:z}$ and has only 3 singular points of type $\frac{1}{2}(1,1,1)$, namely at the solutions of $(g_3 = 0)$.

Family 12: $u = v \prec x \prec y \prec t \prec z$

The 2-ray game of $\mathcal{F}$ is represented in the diagram:

$$\begin{tikzcd} & \mathcal{F} \arrow[r, "\Psi_1"] \arrow[dl, "\Phi"'] & \mathcal{F}_1 \arrow[r, "\Psi_2"] & \mathcal{F}_2 \arrow[dr, "\Phi'"] & \\ \mathbb{P}^1_{u:v} & & & & \mathbb{P}\ (1,1,1,1,2) \end{tikzcd} \qquad ,$$

where Ψ_1 is the anti-flip $(1,1,-1,-3,-2)$ and Ψ_2 is the smooth flip $(1,1,1,-1,-1)$. The singular locus of X is characterised by the coefficient of $t^2 \in f$; this is a cubic in u, v, so for X general $\mathrm{Sing}(X) = 3 \times \frac{1}{2}(1,1,1)$. The map ψ_1, the restriction of Ψ_1 to X, is the Francia anti-flip as the variable t can be eliminated in a neighbourhood of $\Gamma_x = (u_0 : v_0; 1 : 0 : 0 : 0)$ using the monomial x^2t. Similarly, using the monomial xy^3, we can eliminate the variable x in a neighbourhood of the flipping locus of Ψ_2 and observe that ψ_2 is an Atiyah flop. The final map φ', contracts the divisor $(z = 0)$ to a point on an index 1 Fano hypersurface defined by a degree 5 polynomial in $\mathbb{P}(1,1,1,1,2)$. Note that this Fano hypersurface is quasi-smooth

away from the image of contraction, which is a cD_4 singularity as it is locally defined by $x^2 + u^3 + v^3 + y^4$. It was shown in [5] that a general quasi-smooth Fano hypersurface of this type is birationally rigid.

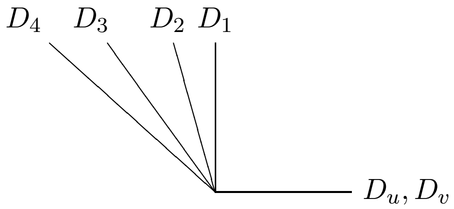
4.2. **Mobile cones.** The aim is to prove that all varieties listed in Table 1 satisfy the conditions of Definition 3.1. In fact the only remaining part to check is the Picard number. This is done in 4.3. On the other hand, we must prove that this is the complete list; meaning any $dP_2/\mathbb{P}^1$ which does not appear in this list cannot have a link to another Mori fibre space following the 2-ray game of $\mathcal{F}$. Therefore we compute various cones of X and $\mathcal{F}$ that we need later.

Proposition 4.4. *Let $\mathcal{F}$ be the toric variety described in 4. Then*

(i) the pseudo-effective cone of $\mathcal{F}$ is generated by D_u and D_4, and
(ii) the mobile cone $\mathrm{Mob}(\mathcal{F})$ *is generated by D_u and D_3,*

where D_u, D_v and D_i are divisors defined by $(u = 0)$, $(v = 0)$ and $(x_i = 0)$.

Proof. The fact that the Picard number of $\mathcal{F}$ is $\rho(\mathcal{F}) = 2$ allows one to write $\mathrm{N}^1(\mathcal{F})_{\mathbb{R}} \cong \mathbb{R}^2$ and hence draw all these cones in the plane



The rays are labelled by divisors that lie on them away from the origin. Note that the rays correspond to some D_i and D_j might coincide. This is exactly when $x_i = x_j$.

Obviously $\langle D_u, \ldots, D_4\rangle \subset \overline{\mathrm{NE}}^1(\mathcal{F})$. We show that any prime divisor corresponding to a lattice point in the plane outside of this cone is not numerically equivalent to an effective divisor. Any divisor given by a lattice point in $\mathbb{R}^2 - \overline{\mathrm{NE}}^1(\mathcal{F})$ is numerically equivalent to a divisor A, A' or A'', where

$$\begin{aligned} A &= -\mu D_u + \lambda D_4 && \text{for } \mu > 0,\ \lambda \geq 0,\\ A' &= -\mu D_u - \lambda D_4 && \text{for } \mu > 0,\ \lambda > 0,\\ A'' &= \mu D_u - \lambda D_4 && \text{for } \mu \geq 0,\ \lambda > 0. \end{aligned}$$

We show that A cannot be effective. Define a curve $l = (x_1 = x_2 = x_3 = 0) \subset \mathcal{F}$, where without loss of generality $b_4 = 1$. We have

$$A \cdot l = -\mu D_u \cdot l + \lambda D_4 \cdot l = -\mu < 0 \qquad .$$

Since A is prime, we must have $l \subset A$. Now consider the family of curves defined by the ideal

$$I_C = (x_1, x_2 + \varphi_{\delta-\beta}(u, v)x_4, x_3\psi_{\delta-\gamma}(u, v)x_4) \quad .$$

For any curve C in this family and any divisor D on $\mathcal{F}$, there exists a positive rational number r such that $r(l \cdot D) = C \cdot D$. Hence The support of this family lies in A. On the other hand, it is easy to see that for any point in D_1 there is a curve C in this family which contains that point. In other words, D_1 is contained in the support of this family and hence $D_1 \subset A$. But A is prime and this is a contradiction.

Proofs for the other two cases, A' and A'' are similar and we do not write them here.

In order to prove (ii), we must show that the cone generated by D_u and D_3 is the $\mathrm{Mob}(\mathcal{F})$. The divisor D_u is mobile as $D_v \in |D_u|$ and hence this linear system is base point free. Any effective divisor $\mathbb{Q}$-linearly equivalent to D_3 is of the form $\lambda D_4 + \mu D_i$ or $\lambda D_4 + \mu D_u$ for some positive integers λ and μ. Therefore $\mathrm{Bs}(D_3) \subset (x_3 = x_4 = 0)$, and hence $|D_3|$ has no fixed component; the fixed part has codimension at least two. This shows that $\langle D_u, D_3\rangle \subset \mathrm{Mob}(\mathcal{F})$. To complete the proof we must show that any effective divisor in $\overline{\mathrm{NE}}^1(\mathcal{F}) - \mathrm{Mob}(\mathcal{F})$ is not mobile. But any such divisor is numerically equivalent to a divisor of the form $\mu D_3 + \lambda D_4$ for some non-negative integers μ and λ. The fixed part of the linear system of such divisor includes D_4 and hence this divisor cannot be mobile. $\square$

Definition 4.5. ([12], Definition 1.10) A normal projective variety X is called a *Mori dream space* if

(i) X is $\mathbb{Q}$-factorial and $\mathrm{Pic}(X) = N^1(X)$ is finitely generated.
(ii) there are finitely many birational maps $f_i \colon X \dashrightarrow X_i$ for $1 \le i \le k$, which are isomorphisms in codimension one, such that if B is a mobile divisor then there is an index $1 \le i \le k$ and a semiample divisor B_i on X_i such that $B = f_i^* B_i$.

The key point of this definition is that it allows one to run MMP on X in a very easy and clear way. If X is a Mori dream space then the pseudo-effective cone $\overline{\mathrm{NE}}^1(X)$ is divided into finitely many rational polyhedra, $R_1, \dots, R_m$,

$$\overline{\mathrm{NE}}^1(X) = \bigcup_{j=1}^{m} R_j \qquad .$$

The mobile cone is a union of $M_1, \dots, M_k$, some subset of the rational polyhedra $R_1, \dots, R_m$, and the birational maps $f_1, \dots, f_k$ defined in 4.5 are precisely the maps φ_{B_i} associated to a big mobile divisor B_i belonging to the interior of each polytope M_i. For details see [12] Proposition 1.11.

It was proved in [2] Corollary 1.3.1 that any log Fano variety is a Mori dream space. In particular, a dP_2 fibration is a Mori dream space. The idea of defining techniques in this article is that we are trying to find dP_2 fibrations $X \subset \mathcal{F}$ whose decomposition of $\mathrm{Mob}(X)$ into $M_1, \dots, M_k$ coincides with the decomposition of $\mathrm{Mob}(\mathcal{F})$ into such polytopes. In other words, X is embedded into $\mathcal{F}$ and

$$\mathrm{Cox}(X) = \mathrm{Cox}(\mathcal{F})/(f = 0) \qquad .$$

Lemma 4.6. *Let $X \subset \mathcal{F}$ be a hypersurface of the rank two toric variety in 4 defined by a homogeneous polynomial of bi-degree $(\omega, 4)$. If X is a dP_2 fibration then $\sigma = \langle L, X \cap D_3 \rangle$ is a subcone of* $\mathrm{Mob}(X)$.

Proof. Similar to the proof of Proposition 4.4 (ii) one can check that $\mathrm{Bs}\,|L|$ is empty and $\mathrm{Bs}\,|D_3|$ has no fixed component. Note that $\mathrm{Bs}\,|D_3|$ is included in the locus $(x_3 = x_4 = 0)$, and this locus must have codimension strictly bigger than 1. Otherwise, if $(x_3 = x_4 = 0)$ defines a divisor on X then Proposition 5.7 implies that X is not a dP_2 fibration. $\square$

4.3. **The Picard group.** The aim in this section is to prove $\mathrm{Pic}(X) \cong \mathbb{Z}^2$ for a general X in Table 1.

Let us first recall some technical tools that we use in the proof. This includes a version of the Lefschetz hyperplane theorem and a generalised Kodaira vanishing theorem.

Theorem 4.7. [Generalised Kodaira vanishing, [16] Theorem 2.70.] *Let (X, Δ) be a proper klt pair. Let N be a $\mathbb{Q}$-Cartier Weil divisor on X such that $N \equiv M + \Delta$, where M is a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Then $H^i(X, \mathcal{O}_X(-N)) = 0$ for $i < \dim X$.*

Remark 4.8. Let V and W be algebraic varieties. Recall that any algebraic map $\pi\colon V \to W$ can be decomposed into finitely many varieties $V_i \subset V$ of varying dimension, on each of which π restricts to a map with constant fibre dimension.

Definition 4.9. [[6] §2.2] Define $D(\pi)$, the *measure of deviation* of $\pi : V \to W$, to be

$$D(\pi) = \sup_i \{(\text{the fibre dimension of } \pi \text{ in } V_i) \; - (\text{the codimension of } V_i \text{ in } V)\} \quad .$$

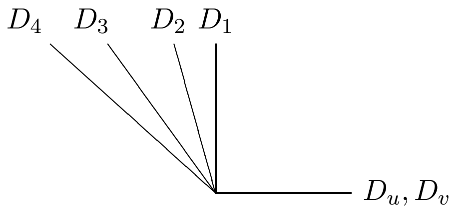
Theorem 4.10. [Lefschetz hyperplane theorem, [6] §2.2] *Let $\pi\colon V \to \mathbb{C}^N$ be a proper map of a purely n-dimensional (possibly singular) algebraic variety into complex affine space. Then $H_i(V) = 0$ for $i > n + D(\pi)$.*

Lemma 4.11. *Let $X \subset \mathcal{F}$ be a hypersurface defined by*

$$\begin{pmatrix} -e \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 0 & 0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix} \quad ,$$

where the variables are in order $u = v \prec\ x_1 \preceq\ x_2 \preceq\ x_3 \preceq\ x_4$ and $\{\beta_1, \beta_2, \beta_3, \beta_4\} = \{1, 1, 1, 2\}$. Suppose $\mathcal{F}_i$ and X_i are birational models of $\mathcal{F}$ and X obtained by small modifications as in Theorem 2.4 and Definition 2.5. Let $\mathcal{U}_i = \mathcal{F}_i - X_i$ be the complement of each X_i in $\mathcal{F}_i$. Consider the point $x = (-e, 4) \in \mathbb{Z}^2$ and recall from Proposition 4.4 that $\mathrm{Mob}(\mathcal{F})$ is a cone in $\mathbb{R}^2 = \mathbb{Z}^2 \otimes \mathbb{R}$ with the same copy of $\mathbb{Z}^2$. If $X \in \mathrm{Int}(\mathrm{Mob}(\mathcal{F}))$, then $H_5(\mathcal{U}_i) = H_6(\mathcal{U}_i) = 0$ for some i.

Proof. Consider the map $\Phi_{|D|}\colon \mathcal{F} \to \mathbb{P}^N$ defined by the linear system of the divisor $D = 4M - eL$ and assume $D \in \mathrm{Mob}(\mathcal{F})$. By Proposition 4.4, $\overline{\mathrm{NE}}^1(\mathcal{F})$ has the following decomposition:



,

where the rays are labelled by divisors that lie on them away from the origin.

From geometric invariant theory we have the following characterisation (possibly after taking a positive multiple of D):

(i) $\Phi_{|D|}$ is an embedding of $\mathcal{F}_i$ if $D \in \mathrm{Int}\,\langle D_i, D_{i+1}\rangle$, where D_i and D_{i+1} do not lie on the same ray.
(ii) $\Phi_{|D|}$ is a small contraction from $\mathcal{F}_i$ if $D = aD_i$ for some positive integer a and $D_i \in \mathrm{Int}(\mathrm{Mob}(\mathcal{F}))$.
(iii) $\Phi_{|D|}$ is an extremal contraction of divisorial or fibre type otherwise.

Suppose $D \in \mathrm{Int}(\mathrm{Mob}(\mathcal{F}))$; in particular it is in one of the cases (i) or (ii) above.

Let $\mathcal{U}_i = \mathcal{F}_i - X_i$, where i is the integer for which (i) or (ii) above is satisfied. Suppose $\varphi\colon \mathcal{U}_i \to \mathbb{C}^N$ be the restriction of $\Phi_{|D|}$ to $\mathcal{U}_i$. The map φ is proper because $\Phi_{|D|}$ is a projective morphism and X_i is the complete preimage of a hyperplane section of the target variety. Since this map contracts at most a 2-dimensional subspace of $\mathcal{F}_i$ and is isomorphism everywhere else, the codimension of every V_j in Definition 4.9 is at least 2, while the fibre dimension is at most 2. Hence $D(\varphi) \le 0$ so by Theorem 4.10 we conclude that $\mathrm{H}_5(\mathcal{U}_i) = \mathrm{H}_6(\mathcal{U}_i) = 0$. Note that $\dim_{\mathbb{C}}(\mathcal{U}_i) = 4$ and $\dim_{\mathbb{R}}(\mathcal{U}_i) = 8$. □

Corollary 4.12. $H^2_c(\mathcal{U}_i) = H^3_c(\mathcal{U}_i) = 0$.

Proof. This follows from Lemma 4.11 and Poincaré duality . □

Lemma 4.13. *Let $\mathcal{F}$ be the ambient toric variety of any family in Table 1 except 1,2 and 3. Then $H^2(\mathcal{F}_i) = \mathbb{Z}^2$ for all models $\mathcal{F}_i$ obtained by flips, flops or antiflips from $\mathcal{F}$.*

Proof. From the short exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_{\mathcal{F}} \to \mathcal{O}^*_{\mathcal{F}} \to 0$$

one constructs the long exact sequence

$$\cdots \to \mathrm{H}^1(\mathcal{F}, \mathbb{Z}) \to \mathrm{H}^1(\mathcal{F}, \mathcal{O}_{\mathcal{F}}) \to \mathrm{H}^1(\mathcal{F}, \mathcal{O}^*_{\mathcal{F}}) \to \mathrm{H}^2(\mathcal{F}, \mathbb{Z}) \to \mathrm{H}^2(\mathcal{F}, \mathcal{O}_{\mathcal{F}}) \to \cdots \quad .$$

On the other hand, for any $\mathcal{F}$ in Families 4,...,12 in Table 1 there exists a birational model $\mathcal{F}_i$, obtained by some flips (flops or antiflips) for which $-K_{\mathcal{F}_i}$ is nef and big. Applying Theorem 4.7

for the pair $(\mathcal{F}_i, 0)$ and divisor $-K_{\mathcal{F}_i}$ gives $\mathrm{H}^j(\mathcal{F}_i, \mathcal{O}_{\mathcal{F}_i}(-K_{\mathcal{F}_i})) = 0$ for all $j < 4$. This vanishing together with Serre duality implies

$$\mathrm{H}^1(\mathcal{F}_i, \mathcal{O}_{\mathcal{F}_i}) = \mathrm{H}^2(\mathcal{F}_i, \mathcal{O}_{\mathcal{F}_i}) = 0 \qquad .$$

The fact that $\mathcal{F}_i$ have rational singularities ensures that the vanishing above holds for all models $\mathcal{F}_i$.

Of course $\mathrm{Pic}(\mathcal{F}_i) \cong \mathbb{Z}^2$ for all models $\mathcal{F}_i$ obtained by flips, flops or antiflips from $\mathcal{F}$. Using the fact that $\mathrm{H}^1(\mathcal{F}_i, \mathcal{O}^*_{\mathcal{F}_i}) \cong \mathrm{Pic}(\mathcal{F}_i)$, the exact sequence above, together with the vanishing statements that we proved imply $\mathrm{H}^2(\mathcal{F}_i) \cong \mathbb{Z}^2$. □

Proposition 4.14. *Let $X \subset \mathcal{F}$ be a hypersurface defined by $f \in H^0(\mathcal{F}, D)$, where $D = 4M - eL$ and $(-e, 4) \in \mathrm{Int}(\mathrm{Mob}(\mathcal{F}))$. If $\mathcal{F}$ is the abient space of one of the families in Table 1 except families 1,2 and 3, then $H^2(X_i) \cong \mathbb{Z}^2$ for $X_i \subset \mathcal{F}_i$, where $\mathcal{F}_i$ is the model specified in Lemma 4.11.*

Proof. Together with Corollary 4.12, the exact sequence

$$\cdots \to \mathrm{H}^2_c(\mathcal{U}_i) \to \mathrm{H}^2(\mathcal{F}_i) \to \mathrm{H}^2(X_i) \to \mathrm{H}^3_c(\mathcal{U}_i) \to \cdots$$

implies $\mathrm{H}^2(\mathcal{F}_i) \cong \mathrm{H}^2(X_i)$. The proof follows from Lemma 4.13. □

Lemma 4.15. *For a general X in Table 1, $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$.*

Proof. For any such X there exists a model X_i obtained by some flips, flops or antiflips from X such that $-K_{X_i}$ is nef and big on X_i. Considering the pair $(X_i, 0)$, which is a klt pair as X_i is terminal, and applying Theorem 4.7 gives $\mathrm{H}^j(X_i, \mathcal{O}_X(-K_{X_i})) = 0$ for all $j < 3$. This together with Serre duality implies $\mathrm{H}^1(X_i, \mathcal{O}_{X_i}) = \mathrm{H}^2(X_i, \mathcal{O}_{X_i}) = 0$. The rationality of singularities of X_i allows one to lift this vanishing to all X_k. In particular, $\mathrm{H}^1(X, \mathcal{O}_X) = \mathrm{H}^2(X, \mathcal{O}_X) = 0$. □

Theorem 4.16. *Let $X \subset \mathcal{F}$ be a general $dP_2/\mathbb{P}^1$ in one of the families in Table 1 then* $\mathrm{Pic}(X) \cong \mathbb{Z}^2$.

Proof. Let X be a general $dP_2/\mathbb{P}^1$ in one of the families of Table 1 except families 1, 2 and 3. By Proposition 4.14, $\mathrm{H}^2(X_i) \cong \mathbb{Z}^2$ for some model X_i obtained by some flips, flops or antiflips from X. On the other hand, Lemma 4.15 implies $\mathrm{H}^1(X_i, \mathcal{O}_{X_i}) = \mathrm{H}^2(X_i, \mathcal{O}_{X_i}) = 0$. Applying this to the exact sequence

$$\cdots \to \mathrm{H}^1(X_i, \mathbb{Z}) \to \mathrm{H}^1(X_i, \mathcal{O}_{X_i}) \to \mathrm{H}^1(X_i, \mathcal{O}^*_{X_i}) \to \mathrm{H}^2(X_i, \mathbb{Z}) \to \mathrm{H}^2(X_i, \mathcal{O}_{X_i}) \to \cdots$$

enables one to see $\mathrm{H}^1(X_i, \mathcal{O}^*_{X_i}) \cong \mathrm{H}^2(X_i, \mathbb{Z})$; hence $\mathrm{Pic}(X_i) \cong \mathbb{Z}^2$. The fact that X_i is isomorphic to X in codimension 1 shows that $\mathrm{Pic}(X) \cong \mathbb{Z}^2$.

In order to finish the proof, we must show that $\mathrm{Pic}(X) \cong \mathbb{Z}^2$ for a general X in families 1, 2 and 3. But we know that any such X is obtained by a blow up of a Fano 3-fold with Picard rank 1, which completes the proof. □

5. Failing cases

In this section, we show that any hypersurface $X \subset \mathcal{F}$ under the hypothesis of Theorem 3.3, which does not appear in the Table 1 either is not a dP_2 fibration or does not provide an $\mathcal{F}$-Sarkisov link.

Let us fix a general setting for $\mathcal{F}$ and X. Let $\mathcal{F}$ be the rank two toric variety with Cox ring $\mathrm{Cox}(\mathcal{F}) = \mathbb{C}[u, v, x_1, x_2, x_3, x_4]$ and irrelevant ideal $I = (u, v) \cap (x_1, \ldots, x_4)$ with the action of $(\mathbb{C}^*)^2$ defined by

$$\begin{pmatrix} 1 & 1 & -a_1 & -a_2 & -a_3 & -a_4 \\ 0 & 0 & b_1 & b_2 & b_3 & b_4 \end{pmatrix}, \tag{4}$$

where a_i are non-negative integers and $\{b_1, \ldots, b_4\} = \{1, 1, 1, 2\}$ such that the coordinate variables of $\mathrm{Cox}(\mathcal{F})$ are in order $u = v \prec x_1 \preceq x_2 \preceq x_3 \preceq x_4$. Let X be a hypersurface of $\mathcal{F}$ defined by a homogeneous polynomial of bi-degree $(\omega, 4)$ with respect to the action above. We sometimes switch these variable names to our favourite u, v, x, y, z, t when we need to write explicit equations. Otherwise, we keep this notation, as it enables us to consider the order of variables without confusion about the position of the variable t and having to divide into three types described at the beginning of Section 4.2. .

5.1. Elimination process.

Here we provide the key tools to eliminate cases which do not occur in Table 1.

In the following lemma, we consider the coordinate variables of $\mathcal{F}$ to be u, v, x, y, z, t and the variable t corresponds to the coordinate, which has been acted by $(\lambda^{-\gamma}, \mu^2) \in (\mathbb{C}^*)^2$.

Lemma 5.1. *If X is taken as a hypersurface in $\mathcal{F}$, it fails to be terminal if any of the following holds:*

(1) $\mathcal{F}$ is of type (i), and $e > 2c$.
(2) $\mathcal{F}$ is of type (ii), and $e > 0$.
(3) $\mathcal{F}$ is of type (iii), and $e > 2$.

Proof. In any of these cases, whenever t appears in a term of f, it is multiplied by a nonconstant polynomial in x, y, z, which implies $\Gamma_t \subset X$. We recall that the curve Γ_t is defined as $\Gamma_t = (x = y = z = 0) \subset X$. Therefore X has a line of singularity, but 3-fold terminal singularities are isolated by [20]. □

We are interested in cases that $\sigma = \mathrm{Mob}(X)$. In particular, these are the cases when the type III and IV 2-ray game of X follows the one from $\mathcal{F}$. The following lemma helps us to eliminate cases when X fails to follow such link at the beginning of the game.

Theorem 5.2. *Let $X \subset \mathcal{F}$ be defined as in 4. If X is not obtained by one of the following, then either it is not a dP_2 fibration or the first step of its 2-ray game cannot be obtained by the restriction of the one from $\mathcal{F}$.*

(i) $a_1 = a_2 = a_3 = a_4 = 0$ *and* $\omega = 1$.

(ii) $a_1 = a_2 = a_3 = 0$*,* $a_4 = 1$ *and* $\omega = 0$.

(iii) $a_1 = a_2 = 0$*,* $a_3 a_4 \neq 0$ *and* $\omega = 0$.

(iv) $x_1 \prec x_2, x_3, x_4$ *and there is a monomial with only variables* x_1, x_2, x_3, x_4 *in the defining equation of* X*.*

Proof. Assume x_1, x_2, x_3, x_4 have equal ratio weight, i.e. $x_1 = x_2 = x_3 = x_4$. Then there is no Ψ_i and the 2-ray game of $\mathcal{F}$ is followed by a fibration to $\mathbb{P}(1,1,1,2)$. Without loss of generality we can assume this common weight is zero. In other words, by adding a multiple of the second row of the matrix A to the first row we can assume $X \subset \mathcal{F}$ is defined by

$$\begin{pmatrix} \omega \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} .$$

If $\omega = 0$, then $X \cong \mathbb{P}^1 \times dP_2$. If we denote the generic fibre by S, then $\mathrm{H}^1(S, \mathcal{O}_S) = 0$ together with Exercise 12.6 in Chapter III [11] implies that $\mathrm{Pic}(X) = \mathrm{Pic}(S) \times \mathrm{Pic}(\mathbb{P}^1)$. And hence $\rho_X > 2$ and therefore X is not a Mori fibre space. If $\omega = 1$, then the equation of X has the form $uf = vg$ for f, g degree 4 homogeneous polynomials in $\mathbb{P}(1,1,1,2)$. It shows that X is the blow up of $\mathbb{P}(1,1,1,2)$ along a curve defined by $(f = g = 0)$. This was done by restricting Φ' to X, which shows the 2-ray game of X comes from $\mathcal{F}$. This case was given as Family 1 in Table 1.
If $\omega > 1$, then X is generically an ω-cover of $\mathbb{P}(1,1,1,2)$, which fails to be a dP_2 fibration.

To move onto the next case, suppose the ratio weight of x_1, x_2, x_3 is equal and normalised to zero and different from that of x_4. In other words, $x_1 = x_2 = x_3 \prec x_4$ and $X \subset \mathcal{F}$ is defined by

$$\begin{pmatrix} \omega \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -a \\ 0 & 0 & b_1 & b_2 & b_3 & b_4 \end{pmatrix} ,$$

for a positive integer a. In this case, the 2-ray game of $\mathcal{F}$ is followed by a divisorial contraction to $\mathbb{P} = \mathrm{Proj} \bigoplus_k \mathrm{Cox}(\mathcal{F})_{(0,k)}$, with exceptional divisor $(x_4 = 0)$. If $\omega < 0$, then X is reducible and hence not a dP_2 fibration.
If $\omega = 0$ and $a = 1$, then φ' is a divisorial contraction from X, which is case (ii). This forms Family 2 and Family 3 in Table 1. The failure of case $\omega = 0$ and $a > 1$ is proved in Lemma 5.6 below.
The interesting case is when $\omega > 0$. In this situation the image of restriction of the contraction on $\mathcal{F}$ to X is a surface, hence this map does not define the 2-ray game of X. This means that X does not have an $\mathcal{F}$-Sarkisov link. But when $b_4 = \omega = a = 1$, we show in Example 5.3 that X is non-rigid. Note that this case does not appear in Table 1 as the 2-ray game is given by a different ambient space. Apart from this special case, if X forms a dP_2 fibration, we expect it to be non-rigid.

For part (iii), assume $a_1 = a_2 = 0$ and $x_1, x_2 \prec x_3, x_4$. In this case, the 2-ray game of $\mathcal{F}$ is continued by an anti-flip (or flop), which contracts $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1$ and extracts a copy of $\mathbb{P}^1 \times \mathbb{P}(a_3, a_4)$. If $\omega = 0$, then the restriction of this operation to X will be a finite number (2 or 4) of disjoint anti-flips (or flops) of type $(1, 1, -a_3, -a_4)$. This is the case mentioned in (iii).
If $\omega < 0$, then the Picard number of X is bigger than two, which is proved in Proposition 5.7. This shows that X is not a dP_2 fibration.
If $\omega > 0$, then the restriction of the ambient anti-flip (flop) defines an small contraction in one side and an isomorphism in the other side, which clearly does not read the 2-ray game of X.

Assume $x_1 \prec x_2, x_3, x_4$. In this case the 2-ray game of $\mathcal{F}$ at the level of Ψ_1 can be read as a flip (flop or anti-flip) of type $(\alpha, \alpha, -\beta_1, -\beta_2, -\beta_3)$. It is obvious that this will restrict to a 3-fold flip (flop or anti-flip) on X if the extracted surface, $\mathbb{P}(\beta_1, \beta_2, \beta_3)$ with coordinate variables x_2, x_3, x_4, intersected with X defines a curve. This will be valid only if this surface is not a subvariety of X. This means the defining polynomial of X must have at least one monomial with only x_i variables. Note that if a term of the form x_1^k appears in the equation, X will pass this step of the 2-ray game isomorphically and nothing contradicts our statements. $\square$

Example 5.3. Let $X \subset \mathcal{F}$ be defined in the usual way by

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -a \\ 0 & 0 & 1 & 1 & 2 & 1 \end{pmatrix} \qquad ,$$

where $a > 0$ is an integer. It was shown in the proof of Theorem 5.2 that such X does not have an $\mathcal{F}$-link. Here we show that X can be embedded into another scroll $\mathcal{F}'$ such that X has an $\mathcal{F}'$-Sarkisov link to another Mori fibre space.

Let us fix the variables of $\mathcal{F}$ in order by u, v, x, y, t, z as usual. The defining polynomial of X is of the form $uf = vg$ for some bi-degree $(0, 4)$ polynomials f, g. Now we apply unprojection operations of [18]. Let s be a rational function defined by

$$s = \frac{f}{v} = \frac{g}{u}$$

with bi-degree $(-1, 4)$. Then treat it as a variable in equations $us = g$ and $vs = f$. This enables us to embed X into the scroll $\mathcal{F}'$:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -1 & -a \\ 0 & 0 & 1 & 1 & 2 & 4 & 1 \end{pmatrix} \qquad ,$$

where the variables in order are $u = v \prec x = y = t \prec s \prec z$. The variety X is embedded into $\mathcal{F}'$ as the complete intersection of two hypersurfaces $us = g$ and $vs = f$.

$\mathcal{F}'$ is a 5-fold toric variety of rank 2 whose 2-ray game starts by an anti-flip (or flop) of type $(1, 1, -1, -a)$ over a surface $\mathbb{P}(1, 1, 2)$. Meaning, it contracts a copy of $\mathbb{P}^1 \times \mathbb{P}(a, a, 2)$ to $\mathbb{P}(1, 1, 1)$ in one side and extracts a copy of $\mathbb{P}(1, a) \times \mathbb{P}(1, 1, 2)$ in the other side. The restriction of this

map to X defines an anti-flip (or flop), consisting 2 disjoint anti-flip (or flop) of type $(1, 1, -1, -a)$. Then it has a divisorial contraction to a codimension 2 Fano 3-fold of index one defined by $Y_{4,4} \subset \mathbb{P}(1, 1, 1, 1, 2, 3)$.

The key point in this example is that the $\sigma \subset \mathrm{Mob}(X)$ but they are not equal. However, as $-K_X$ is still in the pseudo-effective cone, we managed to find another embedding of X for which $\mathrm{Mob}(X)$ is the restriction of that of the ambient space. This allowed us to read $-K_X \in \mathrm{Int}(\mathrm{Mob}(X))$.

Before stating the next lemma, we say a few words about the anticanonical classes of $\mathcal{F}$ and X. By Corollary 2.2.6 in [7] the anticanonical divisor of $\mathcal{F}$ has bi-degree $(2 - \sum a_i, \sum b_i)$. By adjunction we have

$$-K_X = (-K_{\mathcal{F}} - X)|_X$$

and hence the anticanonical divisor of X has bi-degree $(2 - \sum a_i - \omega, 1)$.

Lemma 5.4. *Let X be a hypersurface of $\mathcal{F}$, as in the assumption of Theorem 5.2, satisfying conditions of Theorem 5.2 and Lemma 5.1, which has an $\mathcal{F}$-link. If $-K_X \sim mD_3 - nD_u$ for a positive integer m and a non-negative integer n, then the last morphism of the 2-ray game of X is not an extremal contraction.*

Proof. The proof is given case by case, depending on the ratio weights of the variables. In each case we find a curve inside the exceptional locus of φ', which has positive intersection against the anticanonical class. This shows that the last morphism of the 2-ray game is not an extremal contraction.

Case I $x_2 \prec x_3 \preceq x_4$

Let $C = (x_1 = x_4 = f = 0) \subset \mathrm{Exc}(\varphi')$, where f is the defining polynomial of X. Note that the irrelevant ideal of the domain variety of φ' is defined by $(u, v, x_1, x_2) \cap (x_3, x_4)$. Therefore $D_3 \cdot C = 0$, which implies

$$-K \cdot C = 0 - nD_u \cdot (x_1 = x_4 = f = 0) \leq 0$$

Case II $x_1 \prec x_2 = x_3 \preceq x_4$

Let $C = (x_2 = x_4 = f = 0)$. As the irrelevant ideal in this case is $(u, v, x_1) \cap (x_2, x_3, x_4)$, similar argument shows

$$-K \cdot C = 0 - nD_u \cdot (x_2 = x_4 = f = 0) \leq 0$$

Case III $x_1 = x_2 = x_3 \prec x_4$

The irrelevant ideal in this case is $(u, v) \cap (x_1, x_2, x_3, x_4)$. Without loss of generality we can assume that X is defined by

$$\begin{pmatrix} \omega \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -a \\ 0 & 0 & b_1 & b_2 & b_3 & b_4 \end{pmatrix} \quad ,$$

where a is a positive integer. Theorem 5.2 together with Lemma 5.6 implies $\omega = 0$ and $a = 1$.

□

Remark 5.5. Note that Lemma 5.4 implies that in order to have an $\mathcal{F}$-link from X, it is necessary for the ratio weight of $-K_X$ to be strictly less than that of the coordinate variable x_3. This is simply saying that $-K_X \in \mathrm{Int}(\mathrm{Mob}(X))$.

Lemma 5.6. *Let $X \subset \mathcal{F}$ be defined by*

$$\begin{pmatrix} 0 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -a \\ 0 & 0 & 1 & 1 & 2 & 1 \end{pmatrix} \qquad ,$$

with variables in order $u = v \prec x = y = t \prec z$ with $a \in \mathbb{Z}$, $a \geq 1$. If the integer a is strictly bigger than 1, then the image of the last morphism of the 2-ray game of X is not terminal.

Proof. If $a > 1$, then the image of $\mathcal{F}$ under the last morphism of its 2-ray game is defined by the quotient of $\mathbb{P}(1,1,1,1,2)$ by the action of $\frac{1}{a}(1,1,0,0,0)$. In particular, this variety has a singular locus of dimension 2. Hence the image of X under this map has non-isolated singularities (along a curve) and therefore is not terminal.

□

Proposition 5.7. *Let $X \subset \mathcal{F}$ be defined as before. If $D = (x_3 = x_4 = 0) \subset X$ forms a divisor on X, i.e. if the defining polynomial of X is of the form $x_3 f = x_4 g$, then ρ_X, the Picard number of X, is at least 3.*

Proof. As in the assumption, let the defining polynomial of X be $x_3 f = x_4 g$ for non-constant polynomials f, g. Let $M \sim (x_1 = 0)$ and $L \sim (u = 0)$ be two other divisors on X. We show that D, M and L are linearly independent and hence $\mathrm{Pic}(X)$ has at least three generators. To do so, we find three curves inside X and compute their intersections with these divisors. These number form a 3×3 matrix. If the rank of this matrix is bigger than 3, we have shown that these divisors are linearly independent.

Consider three curves $C_1, C_2, C_3 \subset X$ defined by

$$C_1 = (u = x_3 = x_4 = 0) \qquad C_2 = (x_1 = x_3 = x + 4 = 0) \qquad C_3 = ((v = x_2 = 0) \cap X)$$

Computing intersection numbers gives:

$$\begin{pmatrix} L \cdot C_1 & L \cdot C_2 & L \cdot C_3 \\ M \cdot C_1 & M \cdot C_2 & M \cdot C_3 \\ D \cdot C_1 & D \cdot C_2 & D \cdot C_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & * & 0 \\ * & * & 1 \end{pmatrix} \quad ,$$

where $*$ denotes some numbers that we have no interest in computing them. Which shows that this matrix has full rank and hence $\rho_X > 2$. □

A typical example of a variety concerned in Proposition 5.7 has following shape:

$$X \in \begin{pmatrix} -1 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix}$$

Before we start the next section let us recall that $\mathcal{F}$ is said to be of type (i), (ii) or (iii) if the corresponding action of $(\mathbb{C}^*)^2$ has the following representations. Note that an easy argument shows that any $\mathcal{F}$ considered in this article has a unique representation in one of these types.

$$(i) \qquad A = \begin{pmatrix} 1 & 1 & 0 & -a & -b & -c \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \qquad 0 < c\,,\, 0 \le a \le b$$

$$(ii) \qquad A = \begin{pmatrix} 1 & 1 & -a & -b & -c & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \qquad 0 \le a \le b \le c$$

$$(iii) \qquad A = \begin{pmatrix} 1 & 1 & -a & -b & -c & -1 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \qquad 0 < a \le b \le c \quad ,$$

where a, b and c are non-negative integers and the variables are u, v, x, y, z, t. The conditions on the order of a, b, c imply that in all cases above the variables x, y, z are ordered with $x \preceq y \preceq z$. And if $\mathcal{F}$ is of type (ii) or (iii), then $t \preceq x$.

Table 2 below gathers some computations of the anti-canonical class of $\mathcal{F}$ and X, which we use later.

	Type (i)	Type (ii)	Type (iii)
$-K_{\mathcal{F}}$	$(2-a-b-c)L+4M$	$(2-a-b-c)L+4M$	$(1-a-b-c)L+4M$
$-K_X$	$(2+e-a-b-c)L+M$	$(2+e-a-b-c)L+M$	$(1+e-a-b-c)L+M$

TABLE 2. Anticanonical classes of $\mathcal{F}$ and X

In the next two subsection, we explicitly analyse cases which do not occur in Table 1 and give arguments why each of them fails. Our arguments are based on the materials provided in this part, namely Lemma 5.1, Theorem 5.2, Lemma 5.4 and Proposition 5.7.

5.2. **Hypersurfaces in scrolls of Type** (ii) **or** (iii).

Proposition 5.8. *If $\mathcal{F}$ is of type (iii), then X does not have a link to any other Mori fibre space except for $e = 2, a = b = c = 1$.*

Proof. If $e = 2$, then Lemma 5.4 implies $a + c < 3$, and that means $a = b = c = 1$. Under these numerical conditions a general X passes the first step of the 2-ray game isomorphically and then maps to $\mathbb{P}^2$ with conic fibres. This forms Family 6 in Table 1.
The case $e > 2$ is not concerned, due to Lemma 5.1. For $e < 2$, Lemma 5.4 does the elimination. □

Proposition 5.9. *Suppose $\mathcal{F}$ is of type (ii), and consider its 2-ray game of Type* III *or* IV*. Exactly one of the following cases occurs:*

(1) X does not have an $\mathcal{F}$-link, or

(2) X does have an $\mathcal{F}$-link but it does not lead to an $\mathcal{F}$-Sarkisov link on X, or

(3) X follows the 2-ray game of $\mathcal{F}$ to a Sarkisov link, and we are in one of the cases

(A) $e = a = 0$, $b = c = 1$,

(B) $e = a = b = 0$, $c = 1$,

(C) $e = -1$, $a = b = c = 0$.

Proof. Suppose the given 2-ray game on $\mathcal{F}$ does restrict to a Sarkisov link on X. In particular, X has terminal singularities, so $e \leq 0$ by Lemma 5.1. If $e < 0$, Lemma 5.1 requires $S_k(u,v)t^2 \in f$, where S_k is a general polynomial with variables u, v of degree $-e = k > 0$. The numerology presented in Table 2, shows that $-K_X \sim (2-k-a-b-c)L + M$. This, together with Lemma 5.4, gives the inequality $k + a + c < 2$. But this can be satisfied only if $k = 1$ and $a = b = c = 0$, which is the case (3C).
In the case $e = 0$, a similar argument using the result of Lemma 5.4 forces $a + c < 2$, and this leads immediately to cases (3A,3B) or $e = a = b = c = 0$. but this case gets eliminated by Theorem 5.2. $\square$

In fact, all solutions (3A–3C) provide Sarkisov links when X is general; these are respectively families No. 5, 2 and 1 in Table 1.

5.3. **Families embedded in Type** (i) **scrolls.** Let us recall that the variable with ratio weight zero is fixed to be x throughout this part.
The following lemma forces strong restrictions on f, the defining polynomial of X. It uses the condition on the singularities of X.

Lemma 5.10. *Let $X \subset \mathcal{F}$ be a hypersurface of $\mathcal{F}$ of a Type (i), defined by the polynomial f as*

$$\begin{pmatrix} -e \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & -a & -b & -c \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix},$$

where $a, b, c > 0$. If there is no term of the form $S_d(u,v)x^k l(y,z,t)$ in the equation of f, then X is not terminal, where l is either a linear form on y, z, t or is a constant.

Proof. By Theorem 5.2, f must include at least a monomial with no u or v in it. This already means $e \geq 0$. Let Γ be the curve defined by $(y = z = t = 0)$. If $e = 0$, then $x^4 \in f$ and there is nothing to prove. If $e > 0$, then $\Gamma \subset X$ and in fact by easy computations one could see that $\Gamma \subset \mathrm{Bs}\,|D|$. If there is no term of the form $S_d(u,v)x^k l(y,z,t)$ in f, then X is singular along Γ. In particular, the singular locus of X is not isolated and hence X cannot be terminal. $\square$

If a, b, c are all nonzero, then by Theorem 5.2 f must include at least one pure monomial in the x, y, z, t variables. But this monomial cannot be x^4, as if otherwise holds, then Lemma 5.4 implies

$a + c < 2$ which cannot be satisfied for any pair of positive integers a and c. Hence $abc \neq 0$ implies $e \neq 0$.

On the other hand, if one of a, b, c is zero, then Proposition 5.7 implies $e = 0$. If only two of a, b, c is zero, then irreducibility of X forces $e = 0$. The case $a = b = c = 0$ has been considered in Theorem 3.3.

The following families have already been studied in Theorem 3.3.

$$X \in \begin{pmatrix} 0 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{pmatrix}$$

$$X \in \begin{pmatrix} 0 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{pmatrix}$$

$$X \in \begin{pmatrix} 0 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{pmatrix}$$

Now we consider the families with $e > 0$. We will specify each family by a sequence of positive integers correspond to $(a, b, c; e)$ which represent the following:

$$X \in \begin{pmatrix} -e \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & -a & -b & -c \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix}$$

Note that the columns of the action matrix of $\mathcal{F}$ are not necessarily in order. But the 2-ray game is played each time after considering the appropriate order.
We also introduce two numbers n and κ, which will simplify our notation, by

$$n = a + b + c, \qquad \kappa = 2 + e - a - b - c \qquad .$$

Note that the number κ is associated to the degree of the anticanonical class of X and determines it uniquely as $-K_X \sim \kappa L + M$. Let us recall that L is the divisor linearly equivalent to $(u = 0)$ and M is the one equivalent to $(x = 0)$.
We will be considering every X defined by $(a, b, c; e)$ by varying $n \in \mathbb{N}$ and spot families which link to a different Mori fibre space. The cases $n = 0, 1, 2$ have already been analysed.

- $n = 3$

The only option for $n = 3$ is when $a = b = c = 1$. By Lemma 5.1 $e \leq c$, which can only be satisfied by $e = 1, 2$. The analysis of the case $(1, 1, 1; 1)$ is the Family 7 in Table 1.

A general X defined by $(1, 1, 1; 2)$ is not terminal as it does not not satisfy conditions of Lemma 5.10.

- $n = 4$

This case has only two possibilities: $(1,1,2;e)$ and $(1,2,1;e)$. By Lemma 5.10 we must have $e \leq 2$. If $e < 2$, for both cases X fails to satisfy Lemma 5.4. Remaining cases provide $\mathcal{F}$-Sarkisov links to other Mori fibre spaces. These are Families 8 and 9 in Tables 1.

- $n = 5$

Different partitions of 5 allow us to have $(1,1,3;e)$, $(1,3,1;e)$, $(1,2,2;e)$ or $(2,2,1;e)$. For the first two cases, e cannot be less than 3 as otherwise it fails to fulfil the criteria of Lemma 5.4. It also cannot be more than 3 because of the condition imposed by Lemma 5.10. A similar argument for the other two cases bounds e to be equal to 2.
However, $(1,3,1;3)$ does not have Picard number two by Proposition 5.7. $(1,2,2;2)$ also fails to satisfy Lemma 5.4 condition. The only remaining cases win to provide $\mathcal{F}$-Sarkisov links form Families 10 and 11 in Table 1.

- $n = 6$

Possible partitions of 6 give three candidates $(1,1,4;e)$, $(1,2,3;e)$, $(2,2,2;e)$. Applying numerical conditions imposed by Lemma 5.4, Lemma 5.10 and Proposition 5.7, and running the elimination process, we are left with the $(1,1,4;4)$ and $(1,2,3;3)$. In Lemma 5.11, a reason for failure of $(1,1,4;4)$ is given. The case $(1,2,3;3)$ is precisely the Family 12 in Table 1.

Lemma 5.11. *Let $X \subset \mathcal{F}$ be defined by*

$$\begin{pmatrix} -4 \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & -4 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \qquad ,$$

with variables u,v,x,y,z,t and equation f. Then a general X has Picard number strictly bigger than 2.

Proof. The proof here is the standard method used in Proposition 5.7. The only difference here is that instead of working with X we consider X_1, obtained by flopping a curve in X. Considering the 2-ray game of X restricted from that of $\mathcal{F}$, there is an Atiyah flop on X because we have a term $x^2t \in f$, which allows one to eliminate t in a neighbourhood of Γ_x. As X_1 is obtained by flopping a curve in X, they have isomorphic Picard groups. Hence $\rho_{X_1} > 2$ implies $\rho_X > 2$.

In order to finish the proof, we need to show that there are at least three divisors on X_1, which are linearly independent. We specify three divisors below and then conclude by proving they have non-linearly dependent intersections with three specific curves inside X_1. After a suitable change of coordinates we can assume $f = yz(y-z)(y-\lambda z) + t(x^2+g)$ (for some fixed cross ratio λ), where g is a polynomial of bi-degree $(0,2)$. Setting $t = 0$ in X_1 leaves 4 divisors above the four roots $0, 1, \lambda, \infty$ of the quartic in y,z, each of them a divisor in X_1 isomorphic to $\mathbb{P}^2_{u:v:x}$. Let D be the divisor defined by $(y = 1, z = t = 0)$ and suppose $L \sim (u = 0)$ and $M \sim (x = 0)$ are two other divisors of X_1. We show that these divisors are linearly independent.

Define three curves on X_1 by

$$C_1 = (v = x = f = 0), \quad C_2 = (v = z = f = 0), \quad C_3 = (x = y = t = 0) \qquad .$$

Computing the intersections leads to

$$\begin{pmatrix} C_1.L & C_1.M & C_1.D \\ C_2.L & C_2.M & C_2.D \\ C_3.L & C_3.M & C_3.D \end{pmatrix} = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad .$$

This matrix has full rank and this completes the proof. □

- $n = 7$

Considering different partitions of 7 and applying the numerical elimination process as before, it turns out that there is only one family of three-folds for which a general member is not birationally rigid, which is $(1, 2, 4; 4)$. This forms Family 13 in Table 1.

The following lemma shows that we only need to consider cases where $n \leq 7$.

Lemma 5.12. *Any X with $n > 7$ does not link to any other Mori fibre space by an $\mathcal{F}$-link.*

Proof. Let X be defined by

$$\begin{pmatrix} -e \\ 4 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ 0 & 0 & 1 & \beta_1 & \beta_2 & \beta_3 \end{pmatrix} \qquad ,$$

where $\{\beta_1, \beta_2, \beta_3\} = \{1, 1, 2\}$ and variables are in order $u = v \prec x \preceq x_1 \preceq x_2 \preceq x_3$. Lemma 5.10 implies $e \in \{\alpha_i - m \,|\, 1 \leq i \leq 3\,, \, m = 0, 1\}$. By adjunction $-K_X \sim (2 - m + \alpha_i - \Sigma\alpha_j)L + M$. To fulfil $-K_X \in \mathrm{Int}(\mathrm{Mob}(X))$, the requirement of Lemma 5.4, we must have

$$m + \alpha_1 + \alpha_2 + \alpha_3 - \alpha_i < 2 + \frac{\alpha_2}{\beta_2} \qquad .$$

Proposition 5.7, together with Lemma 5.4 and Theorem 5.2, shows that this inequality has no solution for any choice of m and i. □

6. Cubic surface fibrations over $\mathbb{P}^2$

In this section we consider a similar construction and provide a list of non-rigid families for cubic surface fibrations over $\mathbb{P}^2$. The arguments are very similar and we do not repeat them for this case.

Definition 6.1. A 4-fold cubic fibration over $\mathbb{P}^2$ is a normal, irreducible, projective, complex variety X such that

(a) X is $\mathbb{Q}$-factorial with at worst terminal singularities,
(b) $\mathrm{Pic}\, X \cong \mathbb{Z}^2$,
(c) there exists an extremal morphism of fibre type $\varphi\colon X \to \mathbb{P}^2$, and
(d) the generic fibre of φ is a degree 3 del Pezzo surface.

We denote this by $dP_3/\mathbb{P}^2$.

Let $\mathcal{F}$ be a weighted bundle over $\mathbb{P}^2$ defined by

(i) $\mathrm{Cox}(\mathcal{F}) = \mathbb{C}[u, v, w, x, y, z, t]$,
(ii) $I_{\mathcal{F}} = (u, v, w) \cap (x, y, z, t)$,
(iii) $(\mathbb{C}^*)^2$ action defined by

$$\begin{pmatrix} 1 & 1 & 1 & \alpha & \beta & \gamma & \delta \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} , \tag{5}$$

for $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$.

6.1. **Construction as hypersurfaces.** Without loss of generality we can assume that matrix above is of the form

$$\begin{pmatrix} 1 & 1 & 1 & 0 & -a & -b & -c \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \tag{6}$$

where $a \leq b \leq c$ are non-negative integers. In particular, the variables are in the order $u = v \prec x \preceq y \preceq z \preceq t$.

We denote the basis of $\mathrm{Pic}(\mathbb{F})$ by L , M, with sections $u \in H^0(\mathcal{F}, L)$ and $x \in H^0(\mathcal{F}, M)$. Let $D \in |4M + dL|$ be a divisor in $\mathcal{F}$ for $d \in \mathbb{Z}$ and suppose $X \subset \mathcal{F}$ is a hypersurface defined by $X = (f = 0) \subset \mathcal{F}$ for a general $f \in \mathcal{O}_{\mathcal{F}}(D)$. The aim is to study the birational geometry of those X specified by $(a, b, c; d)$, which satisfy the conditions of Definition 6.1.

6.2. $dP_3/\mathbb{P}^2$ **models.** Here we find those $(a, b, c; d)$ for which the 3-fold X forms a degree 3 del Pezzo surface fibration over $\mathbb{P}^2$, as in Definition 6.1.

Lemma 6.2. *Let $X \subset \mathcal{F}$ be a general hypersurface defined as in 6.1 by sequence of integers $(a, b, c; d)$, where $0 \leq a \leq b \leq c$ and $d > 0$. Then a general X is a $dP_3/\mathbb{P}^2$.*

Proof. If $d > 0$, then the defining polynomial of X is of the form $f = uf_1 + vf_2 + wf_3$ for some polynomials f_i with bidegree $(d-1, 3)$. It implies that the base locus of the linear system $|3M + dL|$ is empty and hence by the Bertini theorem X is smooth. By Theorem 6.12 below, $\mathrm{Pic}(X) \cong \mathbb{Z}^2$ and hence X is a $dP_3/\mathbb{P}^2$. □

Lemma 6.3. *Let $X \subset \mathcal{F}$ be defined by $(a, b, c; 0)$ as before. Then X forms a $dP_3/\mathbb{P}^2$ for any triple (a, b, c) except for $a = b = c = 0$.*

Proof. It is easy to check that for any (a, b, c), the base locus of $|3M|$ is empty and therefore X is smooth. If $a = b = c = 0$, then the Picard number of X is strictly bigger than 2. By Theorem 6.12 $\mathrm{Pic}(X) \cong \mathbb{Z}^2$ for all other cases. □

Lemma 6.4. *Let $X \subset \mathcal{F}$ be a hypersurface defined by $(a, b, c; d)$ as in 6.1, where $0 \leq a \leq b \leq c$ and $d < 0$. Then X is a $dP_3/\mathbb{P}^2$ if*

(i) *the defining polynomial of* X *includes a monomial of the form* $g_k(u, v, w)x^2L(y, z, t)$, *where* g_k *is a homogeneous polynomial in variables* u, v, w *of degree* $k \geq 0$ *and* L *is a linear form in* y, z, t, *and*

(ii) *one of the following holds*

$$d \leq 3a \leq 3b \quad \text{or} \quad d < 3a \leq 3b \qquad .$$

Proof. If $a = b = c = 0$, then $|3M + dL|$ has no sections. If $a = b = 0$ and $c > 0$, then $f = t.g$, hence X is reducible. If only $a = 0$ and $bc \neq 0$, then a similar argument to the one in Proposition 5.7 shows that $\rho_X > 2$.

Let $0 < a \leq b \leq c$ and suppose one of the $d \leq 3a \leq 3b$ or $d < 3a \leq 3b$ holds. Then Theorem 6.12 implies that $\operatorname{Pic}(X) \cong \mathbb{Z}^2$. If $d = 3a = 3b$, then by a similar argument to Lemma 5.11, $\rho_X > 2$ and hence X is not a $dP_3/\mathbb{P}^2$.

Now suppose X is defined such that $0 < a \leq b \leq c$. If the polynomial f has no term of type $g_k(u, v, w)x^2L(y, z, t)$, then a generic point on the surface $S = (y = z = t = 0) \subset X$ has multiplicity at least 2. Therefore X is singular along a 2-dimensional space. Therefore X is not terminal. If f has such a term, then it is either smooth or it is singular only at finitely many points or along a line. □

Combining Lemma 6.2, Lemma 6.3 and Lemma 6.4 enables us to give the following characteristic theorem.

Theorem 6.5. *Let* $X \subset \mathcal{F}$ *be a general hypersurface defined by* $(a, b, c; d)$. *Then one of the following holds:*

(1) *If* $d > 0$, *then* X *is non-singular and satisfies conditions stated in Definition 6.1 .*

(2) *If* $d = 0$, *then* X *is a* dP_3 *fibration by Definition 6.1 for any triple* (a, b, c) *except for* $a = b = 0$, $c > 1$.

(3) $d < 0$ *and*

  (a) $3c < -d$, $|4M + dL|$ *has no sections.*

  (b) $3a \leq 3b < -d \leq 3c$ *and* X *is reducible, hence not a* dP_3 *fibration.*

  (c) $3a < -d \leq 3b \leq 3c$ *and* X *has Picard number* $\rho_X > 2$, *hence does not satisfy conditions of a* dP_3 *fibration.*

  (d) $-d \leq 3a$. *In this case,* X *is a* dP_3 *fibration over* $\mathbb{P}^2$ *only if the equation of* f *has a term of the form* $g_k(u, v, w)x^2L(y, z, t)$ *in it, where* g_k *is a homogeneous polynomial in variables* u, v, w *of degree* $k \geq 0$ *and* L *is linear.*

6.3. $dP_3/\mathbb{P}^2$ **as Mori dream spaces.** In what follows we show that unlike dimension 3, all dP_3 fibrations constructed above have a 2-ray game which is the restriction of that of the ambient space we consider. The idea is based on the following lemma of Kawamata, Matsuda and Matsuki.

Lemma 6.6. ([15] Lemma 5.1.17) *If $\psi\colon X^- \to X^+$ is a flip (flop or antiflip) with exceptional loci $E^- \subset X^-$ and $E^+ \subset X^+$, then the pair* $(\dim E^-, \dim E^+)$ *is exactly one of the pairs*

$$(2,1) \qquad (2,2) \qquad (1,2) \qquad .$$

Theorem 6.7. *Let $X \subset \mathcal{F}$ be a cubic fibration over $\mathbb{P}^2$ obtained from one of the cases in Theorem 6.5. Then the Type* III *or* IV *2-ray game of $\mathcal{F}$ induces the game on X.*

Proof. We prove the theorem case by case on the sign of d and we show that in each case the conditions on the dimension of contracted loci by Lemma 6.6 are satisfied.

Let $d > 0$. If $a > 0$, then the 2-ray game of $\mathcal{F}$ is continued be a flip which restricts to X with dimension pair $(1,2)$. For $a = 0$ and $b > 0$, the situation is $(2,1)$ and for $a = b = 0$ the game finishes by a divisorial contraction or a fibration; Which is fine as far as the 2-ray game of X is concerned.

For $d = 0$, If $a > 0$ then the first step of the game of $\mathcal{F}$ induces an isomorphism on X and the second step is of type $(2,1)$, divisorial contraction or fibration, respectively in cases a, b, $a = b < c$ and $a = b = c$.

If $a = 0$, then the game continues with a $(2,1)$ or divisorial contraction or a fibration exactly as the previous case.

Let $d < 0$. If $a > 0$ then the 2-ray game of $\mathcal{F}$ restricts to X by a $(2,1)$ or $(2,2)$. ☐

Corollary 6.8. *X is a Mori dream space with* $\mathrm{Cox}(X) = \mathrm{Cox}(\mathcal{F})/(f = 0)$. *In particular* $\mathrm{Mob}(X)$ *is generated by L and $D_z = (z = 0)$.*

6.4. **Nonrigid families.** The following arguments eliminate cases that are not going to have an $\mathcal{F}$-link to another Mori fibre space. As a result a list of nonrigid families through their Type III or IV Sarkisov links is given.

Theorem 6.9. *If $-K_X \notin \mathrm{Int}(\mathrm{Mob}(X))$, then the last map of the 2-ray game of X is not extremal.*

Proof. This proof is similar to that of Lemma 5.4. ☐

Lemma 6.10. *If $d < 0$, then $a + k \leq 2$.*

Proof. Using the adjunction formula, one can compute the anticanonical divisor of X as $-K_X \sim (3 + n - a - b - c)L + M$. Theorem 6.9 results in $-K_X \in \mathrm{Int}(\mathrm{Mob}(X))$, which holds if and only if $a + b + c - 3 - d < b$. This implies $a + c < 3 + d$.

On the other hand, from Theorem 6.5 we have $d \leq c - k$. These two inequalities show that $a + k \leq 2$. ☐

Corollary 6.11. $c < 7$.

Proof. Theorem 6.9 implies $a + c < 3 - d$. On the other hand, Theorem 6.7 requires $-d < c$. One can easily check the inequality using these together with Lemma 6.10. ☐

The inequalities above provide upper limits for (a, b, c). Using these and other information provided in this section one can prove that Theorem 6.13 below has the complete list.

Theorem 6.12. *Let $X \subset \mathcal{F}$ be a general $dP_3/\mathbb{P}^2$ as before. If $X \in \mathrm{Int}(\mathrm{Mob}(\mathcal{F}))$, then $\mathrm{Pic}(X) \cong \mathbb{Z}^2$.*

Proof. One can apply same method as in proof of Theorem 4.16 to obtain this result. Note that the proof in this case is much easier as $\mathcal{F}$ and X are smooth. □

Theorem 6.13. *Consider a general hypersurface $X \subset \mathcal{F}$ with*

$$\begin{pmatrix} d \\ 3 \end{pmatrix} \subset \begin{pmatrix} 1 & 1 & 0 & -a & -a & -c \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \quad ,$$

where $0 \leq a \leq b \leq c$. If the the Type III or IV 2-ray game of X leads to another Mori fibre space, then the weights $(a, b, c; d)$ are among those appearing in the left-hand column of Table 3 and Table 4.

The Sarkisov links generated in this way are described in the remaining columns of Tables 3 and 4.

No.	$(a,b,c;d)$	ψ_1	ψ_2	φ'	new model
1	$(0,1,1;1)$	flip	n/a	fibration	$(Y_4 \subset \mathbb{P}^4)/\mathbb{P}^1$
2	$(0,0,1;1)$	n/a	n/a	contraction	Fano $Y_4 \subset \mathbb{P}^5$
3	$(0,0,0;1)$	n/a	n/a	fibration	conic bundle over $\mathbb{P}^3$
4	$(1,1,1;0)$	$\cong$	n/a	fibration	$dP_3/\mathbb{P}^2$
5	$(0,1,1;0)$	$3\times(1,1,1,-1,-1)$ flips	n/a	fibration	$(Y_3 \subset \mathbb{P}^4)/\mathbb{P}^1$
6	$(0,1,2;0)$	$3\times(1,1,1,-1,-2)$ flops	n/a	contraction	$Y_6 \subset \mathbb{P}(1,1,1,1,2,2)$
7	$(0,0,1;0)$	n/a	n/a	contraction	Fano $Y_3 \subset \mathbb{P}^5$
8	$(0,0,2;0)$	n/a	n/a	contraction	Fano $Y_6 \subset \mathbb{P}(1,1,1,2,2,2)$
9	$(0,2,2;0)$	$3\times(1,1,1,-2,-2)$ antiflip	n/a	fibration	$(Y_6 \subset \mathbb{P}(1^3,2^2))/\mathbb{P}^1$
10	$(1,1,1,-1)$	$(1,1,1,-1,-1)$ flip	n/a	fibration	$dP_8/\mathbb{P}^2$
11	$(1,1,2;-1)$	$(1,1,1,-1,-2)$ flop	n/a	contraction	$Y_5 \subset \mathbb{P}(1^5,2)$
12	$(1,1,2;-2)$	$(1,1,1,-1,-1)$ flip	n/a	contraction	$Y_4 \subset \mathbb{P}(1^5,2)$
13	$(1,1,3;-2)$	$(1,1,1,-1,-1,-3;-2)$ flop	n/a	contraction	$Y_7 \subset \mathbb{P}(1^3,2^2,3)$
14	$(1,1,3;-3)$	$(1,1,1,-1,-1)$ flip	n/a	contraction	$Y_5 \subset \mathbb{P}(1^3,2^2,3)$
15	$(1,2,2;-1)$	$(1,1,1,-2,-2)$ antiflip	$(1,1,1,1,-2,-2;2)$ flop	fibration	$(Y_5 \subset \mathbb{P}(1^4,2))/\mathbb{P}^1$
16	$(1,2,2;-2)$	$(1,1,1,-2,-2)$ flop	$(1,1,1,-1,-1)$ flip	fibration	$(Y_4 \subset \mathbb{P}(1^4,2))/\mathbb{P}^1$
17	$(1,1,4;-3)$	$(1,1,1,-1,-1,-4;-3)$ flop	n/a	contraction	$Y_{10} \subset \mathbb{P}(1^3,3^2,4)$
18	$(1,2,3;-3)$	$(1,1,1,-1,-3)$ antiflip	$(1,1,1,-1,-2)$ flop	contraction	$Y_7 \subset \mathbb{P}(1^4,2,3)$
19	$(1,2,3;-3)$	$(1,1,1,-1,-2)$ flop	$\cong$	contraction	$Y_6 \subset \mathbb{P}(1^4,2,3)$

TABLE 3. Part 1 data of Type III and IV links from general degree 3 del Pezzo hypersurface fibrations over $\mathbb{P}^2$

No.	$(a,b,c;d)$	ψ_1	ψ_2	φ'	new model
20	$(2,2,2;-2)$	$(1,1,1,-2,-2)$ antiflip	n/a	fibration	$dP_2/\mathbb{P}^2$
21	$(1,2,4;-3)$	$(1,1,1,-1,-3)$ antiflip	$\cong$	contraction	$Y_9 \subset \mathbb{P}(1^3,2,3,4)$
22	$(1,3,3;-3)$	$(1,1,1,-1,-3)$ antiflip	$\cong$	fibration	$(Y_6 \subset \mathbb{P}(1^3,2,3))/\mathbb{P}^1$
23	$(2,2,3;-3)$	$(1,1,1,-2,-2)$ antiflip	n/a	contraction	$Y_6 \subset \mathbb{P}(1^5,3)$
24	$(1,3,4;-3)$	$(1,1,1,-2,-2)$ antiflip	$\cong$	contraction	$Y_9 \subset \mathbb{P}(1^4,3,4)$
25	$(2,2,4;-4)$	$(1,1,1,-2,-2)$ antiflip	n/a	contraction	$Y_8 \subset \mathbb{P}(1^3,2^2,4)$
26	$(2,3,3;-3)$	$(1,1,1,-2,-3)$ antiflip	$(1,1,1,2,-1,-1;3)$ flop	fibration	$(Y_6 \subset \mathbb{P}(1^4,3))/\mathbb{P}^1$
27	$(1,4,4;-3)$	$(1,1,1,-1,-4,-4;-3)$ antiflip	$\cong$	fibration	$(Y_9 \subset \mathbb{P}(1^3,3,4))/\mathbb{P}^1$
28	$(2,2,5;-5)$	$(1,1,1,-2,-5)$ ntiflip	n/a	contraction	$Y_{10} \subset \mathbb{P}(1^3,3^2,5)$
29	$(2,3,4;-4)$	$(1,1,1,-2,-3)$ antiflip	$(1,1,1,-1,-2)$ flop	contraction	$Y_8 \subset \mathbb{P}(1^4,2,4)$
30	$(2,3,5,-5)$	$(1,1,1,-2,-3)$ antiflip	$(1,1,2,-1,-3)$ flop	contraction	$Y_{10} \subset \mathbb{P}(1^3,2,3,5)$
31	$(2,4,4;-4)$	$(1,1,1,-2,-4)$ antiflip	$(1,1,1,1,-2,-2;2)$ antiflip	fibration	$(Y_8 \subset \mathbb{P}(1^3,2,4))/\mathbb{P}^1$
32	$(2,3,6;-6)$	$(1,1,1,-2,-3)$ antiflip	$\cong$	contraction	$Y_{12} \subset \mathbb{P}(1^3,3,4,6)$
33	$(2,4,5;-5)$	$(1,1,1,-2,-4)$ antiflip	$(1,1,2,-2,-3)$ antiflip	contraction	$Y_{10} \subset \mathbb{P}(1^4,3,5)$
34	$(2,4,6;-6)$	$(1,1,1,-2,-4)$ antiflip	$\cong$	contraction	$Y_{12} \subset \mathbb{P}(1^3,2,4,6)$
35	$(2,5,5;-5)$	$(1,1,1,-2,-5)$ antiflip	$(1,1,2,-3,-3)$ antiflip	fibration	$(Y_{10} \subset \mathbb{P}(1^3,3,5))/\mathbb{P}^1$
36	$(2,5,6;-6)$	$(1,1,1,-2,-5)$ antiflip	$\cong$	contraction	$Y_{12} \subset \mathbb{P}(1^4,4,6)$
37	$(2,6,6;-6)$	$(1,1,1,-2,-6)$ antiflip	$\cong$	fibration	$(Y_{12} \subset \mathbb{P}(1^3,4,6))/\mathbb{P}^1$

TABLE 4. Part 2 data of Type III and IV links from general degree 3 del Pezzo hypersurface fibrations over $\mathbb{P}^2$

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